On the theory of polynomial identities

Antonio Ioppolo

University of L'Aquila
antonio.ioppolo@univaq.it

Young Researchers Algebra Conference 2023
July 25-29, 2023

Algebras

Let F be a fixed field.

Algebra: F-vector space that is also a ring with additional properties relating the 3 operations.

Example

$M_n(F)$ is the algebra of $n \times n$ matrices.

All our algebras are associative!!

Identities

Identity: symbolic expression involving operations and variables which is identically satisfied when the variables are substituted in a given algebraic structure.

- $ab = ba \rightsquigarrow$ commutative identity,
- $a(bc) = (ab)c \rightsquigarrow$ associative identity.

Polynomials

$X = \{x_1, x_2, \ldots\}$ countable set of non-commuting variables.

$F\langle X \rangle$ is the free algebra of polynomials on X over F.

Example

The commutator $[x_1, x_2] = x_1x_2 - x_2x_1 \in F\langle X \rangle$.

Polynomial identities in algebras

Polynomial identities

Definition

A polynomial $f = f(x_1, \dots, x_n) \in F\langle X \rangle$ is a polynomial identity of the algebra A, and we write $f \equiv 0$, if, for any $a_1, \dots, a_n \in A$,

$$f(a_1, \dots, a_n) = 0.$$

PI-algebras: they satisfy at least one non-trivial polynomial identity.

Example

The commutator $[x_1, x_2] \equiv 0$ on any commutative algebra C.

Standard polynomial: $St_m(x_1, \dots, x_m) = \sum_{\sigma \in S_m} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$.

Theorem (Amitsur, Levitzki, 1950)

$St_{2n}(x_1, \dots, x_{2n}) \equiv 0$ *on* $M_n(F)$.

Combinatorial approach in PI-theory.

Identities of an algebra

Problem

Finding all identities satisfied by a given algebra A.

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle : f \equiv 0 \text{ on } A\}.$$

Example (Drensky, 1981)

$\mathrm{Id}(M_2(F)) = \langle St_4, [[x_1, x_2]^2, x_3]\rangle.$

$\mathrm{Id}(M_3(F)) \rightsquigarrow$ no idea !!!

The Specht's problem

Theorem (Kemer, 1987)

If char $F = 0$, $\mathrm{Id}(A)$ *is finitely generated.*

Example (Belov, 2010)

Negative answer in positive characteristic.

Multilinear polynomials

Definition

A polynomial $f = f(x_1, \dots, x_n) \in F\langle X\rangle$ is said to be multilinear if each variable x_i appears exactly once in each monomial of f.

$$P_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} : \sigma \in S_n\}.$$

Example

$St_n = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is a multilinear polynomial.

Analytical approach in PI-theory.

The codimension sequence

Definition

The n-th codimension of an algebra A is the non-negative integer:

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)} \leq n!$$

- **Regev, 1972**: If A is PI, $c_n(A)$ is exponentially bounded.
- **Kemer, 1979**: characterizes alg. with polynomial growth.

The exponent of an algebra

Theorem (Giambruno, Zaicev, 1999)

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{N}.$$

- Positive answer to Amitsur's conjecture ('80).
- One can classify algebras according to $\exp(A)$.

Example (Giambruno, Mishchenko, Zaicev, 2008-2014)

- Families of Lie algebras in which $\exp(A) \notin \mathbb{N}$.
- Families of non-associative algebras s.t. $\exp(A)$ does not exist.

PI-theory techniques

Superalgebras

A superalgebra A is an algebra graded by the cyclic group $\mathbb{Z}_2$:

$$A = A_0 \oplus A_1, \quad \text{with } A_i A_j \subseteq A_{i+j (\text{mod } 2)}.$$

Physics: algebraic structure describing bosons and fermions.

Example

$M_2(F)$ is a superalgebra with grading: $\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\}$.

Step 1: reduction to the finite dimensional case.

Theorem (Kemer, 1988)

For any alg. A, $\mathrm{Id}(A) = \mathrm{Id}(G(B))$, *where* B *is a f.d. superalgebra.*

Step 2: Wedderburn-Malcev decomposition.

Theorem

An algebra A can be decomposed as

$$A = A_1 \oplus \cdots \oplus A_k + J, \text{ where}$$

- *the A_i's are simple algebras,*
- *J is the Jacobson radical of A.*

Simple algebras

Definition

A is simple if $A^2 \neq 0$ and it has no non-zero proper ideals.

Example

The matrix algebra $M_n(F)$ is simple.

Step 3: Classification of simple algebras.

- **Wedderburn-Artin theorems**: just $M_n(F)$.

Last step

Step 4: PI-theory theorems.

Algebras with trace

Definition

Let A be an associative algebra with unit. A linear map $tr : A \to F$ is a trace on A if $tr(ab) = tr(ba)$, for all $a, b \in A$.

Example

The usual trace on the matrix algebra $M_n(F)$ is defined as

$$tr\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = a_{11} + \cdots + a_{nn}.$$

PI-theory for this kind of algebras

A surprising result

Theorem (Razmyslov, 1974, Procesi, 1976)

$\mathrm{Id}(M_n(F), tr)$ *is generated by the n-th Cayley-Hamilton polynomial.*

Example (2×2 matrices)

$\mathrm{Id}(M_2(F), tr) = \langle x_1x_2 + x_2x_1 + tr(x_1)tr(x_2) - tr(x_1)x_2 - tr(x_2)x_1 - tr(x_1x_2) \rangle.$

Recent developments

Let D_n be the algebra of $n \times n$ diagonal matrices.

Example

A trace on D_n is just a linear map $tr : D_n \to F$.

- **I., Koshlukov, La Mattina, 2021**: $\mathrm{Id}(D_2, tr)$ and $\mathrm{Id}(D_3, tr)$.
- **I., Koshlukov, La Mattina, 2021/22**: growth theorems on $c_n^{tr}(A)$.
- **Giambruno, I., La Mattina, 2023**: Amitsur's conjecture.

$$\exp^{tr}(A) = \lim_{n \to \infty} \sqrt[n]{c_n^{tr}(A)} \in \mathbb{N}.$$

Thank you for your attention.