Central endomorphisms of groups and radical rings*

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Central endomorphisms

An endomorphism γ of a group G is called *central* if, for every $x \in G$, the element $x^{-1}x^{\gamma}$ lies in $Z(G)$.

Every central endomorphism of G is *normal* (it commutes with all inner automorphisms of G) and, conversely, a surjective normal endomorphism of G is central. In particular,

$$Aut_c(G) = C_{Aut(G)}(Inn(G)) \trianglelefteq Aut(G).$$

Some examples:

- *Power automorphisms*, i.e. automorphisms of a group G that fix every subgroup of G (Cooper, 1968)
- *Cyclic automorphisms*, i.e. automorphisms γ of a group G such that $\langle x, x^{\gamma} \rangle$ is cyclic for every $x \in G$ (de Giovanni, Newell, Russo, 2014)

A central endomorphism γ of a group G induces an homomorphism

$$f_\gamma \colon x \in G \mapsto x^{-1}x^\gamma \in Z(G)$$

from G to $Z(G)$. Conversely, if h is an homomorphism from G to $Z(G)$, then h defines a central endomorphism γ_h of G given by $x^{\gamma_h} = xx^h$ for every $x \in G$.

Now, let $R = Hom(G, Z(G))$ be the ring of all homomorphisms from G to $Z(G)$, constructed by setting $x^{f+g} = x^f + x^g$ and $x^{fg} = (x^f)^g$ for every $x \in G$ and $f, g \in R$, and consider the *adjoint monoid* R^{ad} of R by defining the *circle operation*

$$x \circ y = x + y + xy$$

for every $x, y \in R$ with 0 as the identity element (Jacobson, 1945).

It is easy to prove that the map

$$f\colon \gamma \in End_c(G) \mapsto f_\gamma \in R$$

is an isomorphism from the monoid $End_c(G)$ of all central endomorphisms of G to R^{ad}. In particular, $Aut_c(G)$ is isomorphic to the *adjoint group* R^0 of R, i.e. the group of units of R^{ad}.

Problem

Investigating conditions under which every central non-zero endomorphism of a group G is an automorphism or, equivalently for the above considerations, $Hom(G, Z(G))$ is a (Jacobson) *radical ring*, i.e. it is a group with the circle operation.

Remark

Let G be a group and suppose that $G = H \times K$ where H, K are normal non-trivial subgroups of G and H is abelian. Then the homomorphism h from G to $Z(G)$ defined by setting

$$x^h = x^{-1} \text{ if } x \in H, x^h = 1 \text{ if } x \in K$$

induces a central endomorphism γ_h of G which is not injective.

In light of this, a group is called *purely non-abelian* if it does not contain abelian non-trivial direct factors.

A counterexample

Let $A = \langle x \rangle \simeq \mathbb{Z}_3, B = \langle y \rangle \simeq \mathbb{Z}$ and define the semidirect product $G = B \ltimes A$ such that $x^y = x^{-1}$ and $[x, B^2] = \{1\}$.

Observe that $Z(G) = B^2$, so that G cannot be a split extension of $Z(G)$. Suppose now that $G = H \times K$ with H abelian non-trivial subgroup of G. Then $H \leq Z(G)$ and, since $G/Z(G) \simeq S_3$, we deduce that K is finite, in particular $Z(K) = \{1\}$ and $Z(G) = H$, a contradiction. So, G is purely non-abelian.

Anyway, since $Hom(G/Z(G), Z(G)) \simeq Hom(S_3, \mathbb{Z}) = \{1\}$, we deduce that $|Aut_c(G)| = 2$ but

$$Hom(G, Z(G)) \simeq Hom(G/G', Z(G)) \simeq Hom(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}.$$

Then there exist central non-zero endomorphisms of G which are not automorphisms.

Some sufficient conditions

Theorem (Adney, Yen, 1965)

Let G be a finite group. Then $Hom(G, Z(G))$ is a radical ring if and only if G is purely non-abelian.

Theorem (Franciosi, de Giovanni, Newell, 1994)

Let G be a purely non-abelian group such that either G/G' or $Z(G)$ is periodic with no infinite abelian sections of prime exponent. Then every central non-zero endomorphism of G is an automorphism.

Some results obtained

We recall that an element e of a ring R is *idempotent* if $e^2 = e$. Observe that if R is radical, then 0 is the only idempotent element of R.

Proposition

Let G be any group. Then G is purely non-abelian if and only if the zero homomorphism is the only idempotent element of the ring $Hom(G, Z(G))$.

Theorem

Let G be a group. Then $Hom(G, Z(G))$ is a radical Artinian ring if and only if G is purely non-abelian and the additive group of $Hom(G, Z(G))$ satisfies the minimal condition on subgroups.

Corollary

Let G be a purely non-abelian group such that the additive group of the ring $Hom(G, Z(G))$ *satisfies the minimal condition on subgroups. Then every central non-zero endomorphism of G is an automorphism. Moreover, the group* $Aut_c(G)$ *of all central automorphisms of G is nilpotent.*

In order to give some examples, we recall that a ring R is a *zero-ring* (in the sense of Baer) if it has the *zero multiplication*, i.e. $xy = 0$ for every $x, y \in R$. Clearly, if G is a group such that $Hom(G, Z(G))$ is a radical zero-ring, then $Aut_c(G)$ is isomorphic to the additive group of $Hom(G, Z(G))$.

Example 1

Let P and Q be groups of type 2^∞ with generators $1 = a_0, a_1, \dots$ and $1 = b_0, b_1, \dots$ such that $a_{n+1}^2 = a_n, b_{n+1}^2 = b_n$ for each non-negative integer n, respectively. Then, define the automorphisms α and β of $X = P \times Q$ such that

$$a_n^\alpha = a_n^3,\ a_n^\beta = a_n b_n,\ b_n^\alpha = b_n^\beta = b_n$$

for each non-negative integer n. Put $Y = \langle \alpha, \beta \rangle$ and $G = Y \ltimes X$.

Proposition

G is a purely non-abelian group such that $Hom(G, Z(G))$ is a zero-ring whose additive group is an infinite abelian group satisfying the minimal condition on subgroups.

Example 2

Let $H = \langle x \rangle$ and $K = \langle y \rangle$ be cyclic subgroups of orders p^4 and p^3 with p an odd prime, respectively. Consider the semidirect product

$$G = K \ltimes H$$

where $x^y = x^{1+p^3}$.

Proposition

G is a finite purely non-abelian group such that $Hom(G, Z(G))$ is not a zero-ring and $Aut_c(G)$ is not abelian.

An application to the Yang-Baxter equation

Let X be a non-empty set and

$$r\colon X^2 \to X^2$$

be a bijective map. Written $r(x,y) = (\sigma_x(y), \sigma_y(x))$ for suitable maps $\sigma_x, \sigma_y\colon X \to X$, we recall that the pair (X, r) is a *set-theoretical solution* of the Yang-Baxter equation if

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

where

$$r_1\colon r \times id_X\colon X^3 \to X^3$$

and

$$r_2\colon id_X \times r\colon X^3 \to X^3.$$

Moreover, the solution is said to be *non-degenerate* and *involutive* if the maps σ_x, σ_y are bijective and $r^2 = id_{X^2}$.

Now, let G be a group in which $R = Hom(G, Z(G))$ is a radical ring, so that the map

$$\rho_g \colon f \in R \mapsto fg + f \in R$$

defined for every $g \in R$, is bijective since R is a group with the circle operation. Applying a result by [F. Cedó, E. Jespers, J. Okniński: "Braces and the Yang-Baxter equation", *Comm. Math. Phys.* 327 (2014), 101-116.]), we deduce that the map

$$r \colon R^2 \to R^2$$

defined by $r(f,g) = \left(\rho^{-1}_{\rho_g(f)}(g), \rho_g(f)\right)$ for every $f, g \in R$, is a non-degenerate involutive set-theoretical solution of the Yang-Baxter equation.

Taking into account the p-group G of the Example 2, one can implement the above procedure in GAP system computing a solution of the Yang-Baxter equation for small primes p. In particular, since

$$Hom(G, Z(G)) \simeq \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$$

we have 59049 central automorphisms of G for $p = 3$.

Many Thanks!