On Quadratic Rational Groups

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Introduction

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2. Related to "nice" characterization for central units of $\mathcal{U}(\mathbb{Z}G)$.
3. Nice bound of the spectra in case of solvable quadratic rational groups, important to study Gruenberg-Kegel graphs.

Notation

- Every group is finite.
- $x \sim y$ denotes the conjugation in the group.
- $|g|$ is the order of the element $g \in G$.
- $B_G(g) := \frac{N_G(<g>)}{C_G(<g>)}$
- $\mathbb{Q}_n := \mathbb{Q}(e^{2\pi i/n})$

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g)|g \in G)$.

Definition

An element $x \in G$ is called **semirational** iff $\exists m_x$ such that for every $(j, |x|) = 1$ then $x^j \sim x$ or $x^j \sim x^{m_x}$. A group is called **semirational** iff every element is semirational. A group is called **r-semirational** if for any $x \in G$ $m_x = r$. In particular a group is called **inverse-semirational** if is $-1-$semirational.

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- D_{10} is ± 3-semirational but not inverse semirational.
- `SmallGroup(32, 42)` is quadratic rational but not semirational.
- `SmallGroup(32, 9)` is semirational but not quadratic rational.

Character Table Duality

Cut groups

Lemma

Let G be a finite quadratic rational group. Then the group of central units of $\mathbb{Z}G$ is finitely generated and the number of generator is exactly the number of irreducible character with real quadratic extension.

In general we have the inclusion

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \geq \pm\mathcal{Z}(G)$$

but there is a family of groups that satisfies the following equality:

Definition

A finite group G is called **cut** (central units trivial) iff

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm\mathcal{Z}(G)$$

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

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3. For any $x \in G$ then either $|B_G(x)| = \varphi(|x|)$ or $|B_G(x)| = \varphi(|x|)/2$ and $x \not\sim x^{-1}$.
4. If $\mathbb{Q}G \cong \bigoplus_{k=1}^{m} M_{n_k}(D_k)$ is the Wedderburn decomposition where $m, n_k \in \mathbb{Z}_{\geq 1}$ and D_k rational division algebras for each k, then

$$\mathcal{Z}(D_k) \cong \mathbb{Q}(\sqrt{-d})$$

for some $d \in \mathbb{Z}_{\geq 0}$ square free.

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for some $d \in \mathbb{Z}_{\geq 0}$ square free.
5. For each $\chi \in \mathrm{Irr}(G)$, the field of values of χ is $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{Z}_{\geq 1}$, i.e. has field of values equal to $\mathbb{Q}$ or an immaginary quadratic extension of $\mathbb{Q}$.

Isomorphism Problem and Prime Graph Question

One of the main motivation for studying the finite subgroups of $\mathcal{U}(\mathbb{Z}G)$ is the following problem that asks whether the ring structure of RG determines the group G up to isomorphism:

Isomorphism Problem (ISO)

Does the group rings RG and RH being isomorphic imply that so are the groups G and H? In particular for $R = \mathbb{Z}$.

A weaker answer to the Isomorphism problem could be provided by solving the following problem:

Prime graph Question (PQ)

Let G be a finite group. Do G and $V(\mathbb{Z}G)$ have the same prime graph?

This problem is particularly interesting since there exist a reduction theorem.

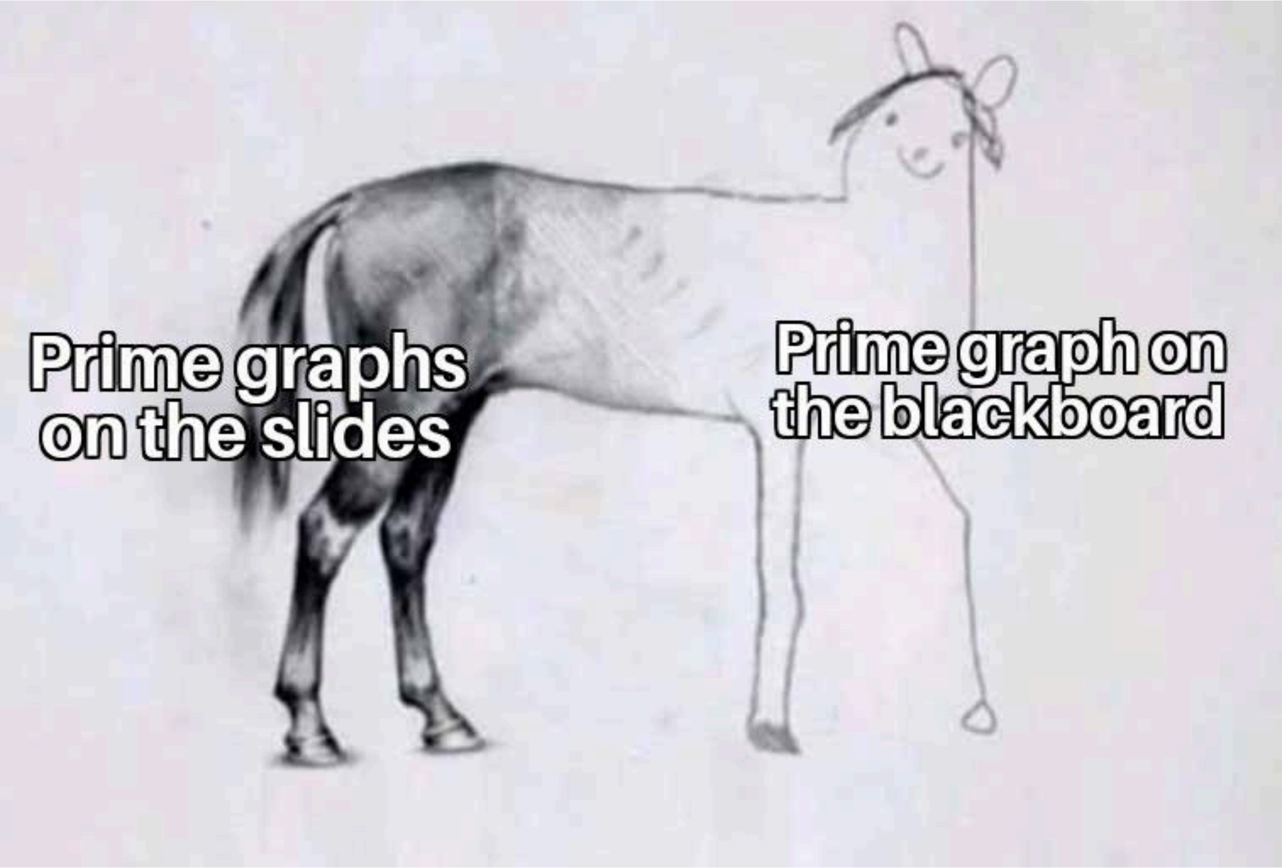
Gruenberg-Kegel Graph (Prime graph)

To every finite group G we can attach a graph that is related to the prime spectra of G.

Definition

The prime graph (or Gruenberg-Kegel graph) is the undirected loop-free and multiple-free graph whose vertices are the primes in the prime spectra of G, and two vertices p and q are joined by an edge, if and only if G contains an element of order pq.

Examples of prime graphs



Results on prime graphs of solvable groups

We can see that in the case of solvable groups there is a nice theorem that characterize the prime graph.

Theorem (Gruenberg-Kegel)

Let G a finite solvable group, then $\Gamma_{GK}(G)$ has at most 2 connected components, and has exactly 2 components, if and only if G is a Frobenius group or a 2-Frobenius group.

In particular if the group G is solvable then the prime graph question holds. Still interesting is to understand what kind of graphs can be realized by a (certain) group.

Results on prime graphs of cut groups

Since there is a reduction theorem that allows us to consider the Prime graph Problem only on almost simple group then we have the following theorems:

Theorem H (Bächle, Kiefer,Maheshwary, Del Rio)

The Prime graph question has a positive answer for finite rational groups.

Theorem G (Bächle, Kiefer,Maheshwary, Del Rio)

Let g be a finite cut group such that there is no epimorphism $G \to \mathbb{M}$, where $\mathbb{M}$ denotes the sporadic simple Monster group. Then the Prime graph Question has a positive answer for G.

What can we say about quadratic rational groups?

Solvable case

Solvable case

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Let G be a solvable quadratic rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}$$

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If the group is inverse-semirational then

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A group is called **quasi-rational** if there exists $r \in \mathbb{Z}$ such that $(r, exp(G)) = 1$ and G is r-semirational. A group is called **2-quasi-rational** if is r-semirational with $|r| = 2$ in $(\mathbb{Z}/exp(G)\mathbb{Z})^{\times}$.

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Theorem (MV)

Let G be a solvable 2-quasi-rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7\}$$

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1. G is r-semirational.
2. $\forall \chi \in \mathrm{Irr}(G)$ $\mathbb{Q}(\chi)$ is quadratic or rational and $\mathbb{Q}(\chi) \cap \mathbb{F} = \mathbb{Q}$.

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$$R_G := \{r \in (\mathbb{Z}/n\mathbb{Z})^{\times} | G \text{ is } r - semirational\}$$

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We can observe that, fixed the group G, R_G is the lateral of the group:

$$H_G = \{r \in (\mathbb{Z}/n\mathbb{Z})^\times | g^r \sim g \; \forall g \in G\} \cong \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q}(G))$$

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Table: Possible R_G for quasi-rational 2-groups with exponent at least 8

$\{-1,3\}$	$\{-1,-3\}$	$\{3,-3\}$	$\{-1\}$	$\{3\}$	$\{-3\}$
$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^{-3}$	$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^3$	$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x\rangle_2$ $a^x = a^3$ $b^x = a^4b^{-1}$ $a^y = a^5b^2$ $b^y = b^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x,y\rangle_2$ $a^x = a^{-1}$ $b^x = a^4b^{-1}$ $a^y = a^5b^2$ $b^y = b^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x,y\rangle_2$ $a^x = a^{-1}$ $b^x = a^4b^{-1}$ $a^y = a^5b^2$ $b^y = a^4b$

$\{2,3\}-$groups

$\{\pm5,\pm7\}$	$\{\pm7,\pm11\}$	$\{\pm5,\pm11\}$
$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{-1}$ $a^y=a^{-11}$ SmallGroup(96,115)	$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{-1}$ $a^y=a^{-5}$ SmallGroup(96,121)	$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{-1}$ $a^y=a^{7}$ SmallGroup(96,117)
$\{-1,-7,5,11\}$	$\{-1,-11,5,7\}$	$\{-1,-5,7,11\}$
$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{-11}$ $a^y=a^{-5}$ SmallGroup(96,183)	$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{-5}$ $a^y=a^{11}$ SmallGroup(96,120)	$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{5}$ $a^y=a^{-11}$ SmallGroup(96,113)
	$\{-1,-11,-5,-7\}$	
	$\langle a\rangle_{24}:\langle x,y\rangle_2$ $a^x=a^{5}$ $a^y=a^{11}$ SmallGroup(96,118)	
$\{-1,11\}$	$\{7,-5\}$	$\{7,11\}$
SmallGroup(192,95)	SmallGroup(192,305)	SmallGroup(192,412)
$\{5,7\}$	$\{-1,-7\}$	$\{\pm7\}$
SmallGroup(192,414)	SmallGroup(192,713)	SmallGroup(192,415)
$\{-1,7\}$	$\{-7,-5\}$	$\{5,-7\}$
SmallGroup(192,418)	SmallGroup(192,435)	SmallGroup(192,623)
$\{-1,-5\}$	$\{\pm5\}$	$\{11,-5\}$
SmallGroup(192,440)	SmallGroup(192,949)	SmallGroup(192,438)
$\{-1,5\}$	$\{5,11\}$	$\{11,-7\}$
SmallGroup(192,1396)	SmallGroup(192,632)	SmallGroup(192,726)
$\{7\}$	$\{-5\}$	$\{-1\}$
SmallGroup(192,424)	SmallGroup(192,445)	SmallGroup(192,634)
$\{5\}$	$\{11\}$	$\{-11\}$
SmallGroup(192,595)	SmallGroup(192,631)	?


Thanks
for your
Attention


When you add a meme after acknowledgments:
I don't know. I never thought I'd get this far