Bi-Skew Braces and Hopf–Galois structures with a surjective Hopf–Galois correspondence

Joint work with Lorenzo Stefanello

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Skew braces

Definition ([GV17])

A skew (left) brace is a triple $(A, +, \circ)$ where

- A is a set,
- $(A, +)$ and $(A, \circ)$ are groups,
- for all $a, b, c \in A$, the equality

$$a \circ (b + c) = (a \circ b) - a + (a \circ c),$$

$(A, +)$ is called the additive group and $(A, \circ)$ the multiplicative group.

λ-map

Given a skew brace $(A, +, \circ)$ and $a, b \in A$, we define

$$\lambda_a(b) = -a + a \circ b.$$

Then we obtain a group homomorphism

$$\lambda : (A, \circ) \to \mathrm{Aut}(A, +), \quad a \mapsto \lambda_a.$$

Definition

A left ideal of a skew brace $(A, +, \circ)$ is a subgroup I of $(A, +)$ such that $\lambda_a(I) = I$ for all $a \in A$. It is automatically also a subgroup of $(A, \circ)$.

Bi-skew braces

Definition ([Chi19])

A skew brace $(A, +, \circ)$ is a bi-skew brace if also $(A, \circ, +)$ is a skew brace.

Easy examples

Let $(A, \circ)$ be a group.

- The trivial skew brace $(A, \circ, \circ)$ is a bi-skew brace.
- The almost trivial skew brace $(A, \circ_{\mathrm{op}}, \circ)$ is a bi-skew brace.

Why bi-skew braces?

- A finite skew brace $(A, +, \circ)$ yields a Hopf-Galois structure of type $(A, +)$ on a Galois extensions with Galois group isomorphic to $(A, \circ)$. So for a given multiplicative group, how do we construct an additive group?
- Most constructions: additive group $\to$ multiplicative group.

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- A finite skew brace $(A, +, \circ)$ yields a Hopf-Galois structure of type $(A, +)$ on a Galois extensions with Galois group isomorphic to $(A, \circ)$. So for a given multiplicative group, how do we construct an additive group?
- Most constructions: additive group $\to$ multiplicative group.
- No distinction for bi-skew braces!

Bi-skew braces

Theorem ([Car20])

Let $(A, +, \circ)$ *be a skew brace, the following properties are equivalent:*

1. A *is a bi-skew brace.*
2. $\lambda : (A, \circ) \to \mathrm{Aut}(A, \circ)$ *is a homomorphism.*
3. $\lambda : (A, +) \to \mathrm{Aut}(A, +)$ *is an anti-homomorphism.*
4. $\lambda_a \lambda_b \lambda_a^{-1} = \lambda_{\lambda_a(b)}$ *for all* $a, b \in A$.

Brace blocks

Definition ([Koc21])

Let A be a set. A brace block, denoted by $((A, \circ_i) \mid i \in I)$, consists of a family of group operations $((A, \circ_i) \mid i \in I)$ on A such that for all $i, j \in I$, $(A, \circ_i, \circ_j)$ is a skew brace.

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Clearly all skew braces $(A, \circ_i, \circ_j)$ must be bi-skew braces.

Brace blocks

Example

Given a bi-skew brace $(A, +, \circ)$, we have a brace block $((A, +), (A, \circ))$.

General construction

Theorem ([ST23a])

Let $(\lambda_i : (A,+) \to \mathrm{Aut}(A,+) \mid i \in I)$ *be anti-homomorphisms. Then the following are equivalent:*

1. *The family* $(A,+) \cup ((A,\circ_i) \mid i \in I)$*, where* $a \circ_i b = a + {}^{\lambda_i(a)}b$*, forms a brace block.*
2. *For all* $\alpha, \beta \in (\lambda_i : (A,+) \to \mathrm{Aut}(A,+) \mid i \in I)$ *and* $a, b \in A$*,*

$$\alpha_a \beta_b \alpha_a^{-1} = \beta_{\alpha_a(b)}. \tag{1}$$

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Is this actually useful to construct brace blocks?

A more manageable construction

If all α_a and β_b commute, then (1) becomes

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Theorem ([ST23a])

Let $(A,+)$ be a group, let M be an abelian subgroup of $\mathrm{Aut}(A,+)$, and let $\mathcal{S}$ be the set of group (anti-)homomorphisms $\lambda\colon A \to M$ such that $\lambda_{\psi(a)} = \lambda_a$ for all $a \in A$ and $\psi \in M$. Then $((A, \circ_\lambda) \mid \lambda \in \mathcal{S})$ is a brace block, where

$$a \circ_\lambda b = a + \lambda_a(b).$$

A more manageable construction

Is it too restrictive to ask that all λ map into abelian subgroup?

A more manageable construction

Is it too restrictive to ask that all λ map into abelian subgroup?
Not at all! All known constructions of brace blocks [BNY23, CS21, CS22, Koc22] satisfy this property.

A more manageable construction

The question remains: how can we find abelian subgroups M of $\mathrm{Aut}(A,+)$ and maps λ satisfying $\lambda_{\psi(a)} = \lambda_a$ for all $a \in A$, $\psi \in M$?

Our intermediate construction with inner automorphisms

Choose an abelian $M \leq \mathrm{Inn}(A,+)$ and let $\mathcal{S}$ be the group homomorphisms $\lambda : A \to M$ such that $\lambda_{\psi(a)} = \lambda_a$ for all $\psi \in M$.

Our intermediate construction with inner automorphisms

Choose an abelian $M \leq \mathrm{Inn}(A,+)$ and let $\mathcal{S}$ be the group homomorphisms $\lambda : A \to M$ such that $\lambda_{\psi(a)} = \lambda_a$ for all $\psi \in M$. As for any ψ, there exists $b \in A$ such that $\psi(a) = b + a - b$, we find for any $\lambda : A \to M$ that

$$\lambda_{\psi(a)} = \lambda_{b+a-b} = \lambda_b \lambda_a \lambda_b^{-1} = \lambda_a.$$

Corollary ([CS22])

Let $(A,+)$ be a group, let M be an abelian subgroup of $\mathrm{Inn}(A,+)$, and let $\mathcal{S} = \mathrm{Hom}(A,M)$. Then $((A,\circ_\lambda) \mid \lambda \in \mathcal{S})$ is a brace block, where

$$a \circ_\lambda b = a + \lambda_a(b).$$

Surjective Hopf-Galois correspondence

Proposition ([ST23b])

Let L/K be a Galois extension with Galois group $(A, \circ)$. For a skew brace $(A, +, \circ)$ and its related Hopf-Galois structure H, we have a bijective correspondence

$$\{\text{left ideals of } (A, +, \circ)\}$$
$$\updownarrow$$
$$\{\text{intermediate fields of } L/K \text{ in the Hopf-Galois correspondence}\}$$

In light of the usual Galois correspondence, if we want the Hopf-Galois correspondence to be surjective, we need that every subgroup of $(A, \circ)$ is a left ideal of $(A, +, \circ)$.

Skew braces with surjective HG-correspondence

Recall that for a bi-skew brace $(A, +, \circ)$, $\lambda_a \in \mathrm{Aut}(A, \circ)$ for all $a \in A$. So we are interested in automorphisms of $(A, \circ)$ that map every subgroup to itself: these are called power automorphisms.

Skew braces with surjective HG-correspondence

For a group G, we define the norm $N(G)$ as the intersection of the normalizers of all subgroups of (G). Then the inner automorphisms coming from elements in $N(G)$ are precisely the inner power automorphisms. By [Sch60] we know that $N(G)$ is contained in the second center of G.

Bi-skew braces with surjective correspondence

Corollary ([CS22])

Let $(A, \circ)$ be a group, let M be an abelian subgroup of $\mathrm{Inn}(A, \circ)$, and let $\mathcal{S} = \mathrm{Hom}(A, M)$. Then $((A, \circ_\lambda) \mid \lambda \in \mathcal{S})$ is a brace block, where

$$a +_\lambda b = a \circ \lambda_a(b).$$

By taking $M = N(A, \circ)/Z(A, \circ) \subseteq \mathrm{Inn}(A, \circ)$ we find a brace block $((A, +_\lambda) \mid \lambda \in \mathrm{Hom}(A, M))$. So in particular we obtain bi-skew braces $(A, +_\lambda, \circ)$ yielding a surjective HG-correspondence.

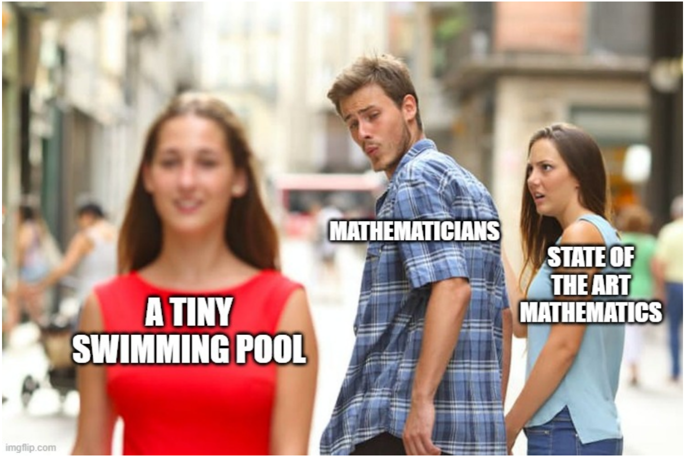
An example

Let $(A, \circ) = Q_8$. Every subgroup of $(A, \circ)$ is normal, so $N(A, \circ) = A$. Hence $M = N(A, \circ)/Z(A, \circ) = A/Z(A, \circ) \cong C_2^2$. It follows that

$$|\mathcal{S}| = |\mathrm{Hom}(A, M)| = |\mathrm{Hom}(C_2^2, C_2^2)| = |M_2(\mathbb{F}_2)| = 16$$

We obtain a brace block $((A, +_i) \mid i \in \{1, ..., 16\})$ such that for all i, $(A, +_i, \circ)$ yields a surjective HG-correspondence on Galois extensions with Galois group Q_8.

The end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