On the algebraic structure of dual weak braces and the Yang-Baxter equation

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UNIVERSITÀ DEL SALENTO
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Aims

This talk aims to:

- Introduce the algebraic structure of the *dual weak-brace*;
- Analyse set-theoretic solutions of the Yang-Baxter equation associated to any dual weak braces;
- Focus on some structural properties of dual weak braces.

Set-theoretic solutions of the Yang-Baxter equation

The quantum Yang-Baxter equation is a basic equation of the statistical mechanics that arose from Yang's work in 1967 and Baxter's one in 1972.

G. Drinfel'd, *On some unsolved problems in quantum group theory, in: Quantum Groups*, Leningrad, 1990, in: Lecture Notes in Math. vol.**1510**(2) Springer, Berlin,(1992), 1–8.

If B is a non-empty set, a *set-theoretic solution* of the Yang-Baxter equation is a map $r : B \times B \to B \times B$ that satisfies the *braid equation*, i.e.,

$$(r \times \mathrm{id}_B)(\mathrm{id}_B \times r)(r \times \mathrm{id}_B) = (\mathrm{id}_B \times r)(r \times \mathrm{id}_B)(\mathrm{id}_B \times r).$$

The Drinfel'd challenge

Determine all the set-theoretic solutions of the Yang-Baxter equation.

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Determine all the set-theoretic solutions of the Yang-Baxter equation.

Notation and terminology

Hereinafter, we will shortly call *solution* any set-theoretic solution to the Yang-Baxter equation. Moreover, if r is a solution on B, for all $a, b \in B$, we define the maps $\lambda_a, \rho_b : B \to B$ and write the map r as

$$r(a,b) = (\lambda_a(b), \rho_b(a)).$$

A solution r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in B$.
- *right non-degenerate* if ρ_b is bijective, for every $b \in B$.
- *non-degenerate* if r is both left and right non-degenerate.
- *involutive* if $r^2 = \mathrm{id}_{B\times B}$.
- *idempotent* if $r^2 = r$.

- The identity map $\mathrm{id}_{B\times B}$ is involutive, idempotent, and degenerate.
- The twist map τ is involutive and non-degenerate.

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Rump's approach

In 2007, Rump traced a novel research direction in the study of solutions.

We recall that if $(B, +, \cdot)$ is a ring and $\circ$ the *adjoint operation* on B, i.e.,

$$\forall\ a, b \in B \quad a \circ b = a + b + a \cdot b,$$

then B is said to be *Jacobson radical* if $(B, \circ)$ is a group (with identity 0).

Any Jacobson radical ring gives rise to a solution $r : B \times B \to B \times B$ defined by

$$r\,(a, b) := \left(\lambda_a\,(b)\,,\ \lambda^{-1}_{\lambda_a(b)}\,(a)\right)$$

where $\lambda_a\,(b) := a \cdot b + b$, for all $a, b \in B$. In particular, r is non-degenerate and involutive.

More generally, non-degenerate involutive solutions are strictly related to the structure of *braces*. Even more generally, non-degenerate bijective solutions can be obtained through *skew braces*.

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Skew braces

Definition (Rump - 2007; Guarnieri, Vendramin - 2017; Cedó, Jespers, Okniński - 2014)

A triple $(B, +, \circ)$ is said to be a *skew brace* if $(B, +)$ and $(B, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds, for all $a, b, c \in B$. If $(B, +)$ is abelian then B is a *brace*.

The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.

- If $(B, +)$ is a group, then $(B, +, +)$ and $(B, +, +^{op})$ are skew braces called the *trivial* and the *almost trivial skew brace* on $(B, +)$, respectively.
- Any Jacobson radical ring is a brace. Indeed, if $(B, +, \cdot)$ is a Jacobson radical ring, then $(B, +, \circ)$ is a brace with $\circ$ is the adjoint operation, where $\mathbf{a} \circ \mathbf{b} := \mathbf{a} + \mathbf{b} + \mathbf{a} \cdot \mathbf{b}$, for all $a, b \in B$.
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Solutions associated to skew braces

Theorem (Rump - 2007; Guarnieri, Vendramin - 2017)

If B is a skew brace, then the map $r_B : B \times B \to B \times B$ defined by

$$r_B(a,\, b) := \left(-a + a \circ b,\; (-a + a \circ b)^{-} \circ a \circ b\right)$$

is a non-degenerate bijective solution (with a^{-} the inverse of a with respect to $\circ$, for every $a \in B$).

Remark: r_B is involutive $\iff$ $(B, +, \circ)$ is a brace.

Theorem (Smoktunowicz, Vendramin - 2018)

*If B is a **finite** skew brace, then the solution associated to B is such that*

$$r_B^{2n} = \mathrm{id}_B$$

where $n \in \mathbb{N}$ is the exponent of the additive quotient group $B/Z(B)$.

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The inverse map r_B^{-1}

If $(B, +, \circ)$ is a skew brace, then the structure

$$B^{op} := (B, +^{op}, \circ)$$

with $a +^{op} b := b + a$, for all $a, b \in B$, is a skew brace called the *opposite skew brace of* $(B, +, \circ)$.

[Koch, Truman - 2020]

The solution

$$r_{B^{op}}(a, b) = \left(a \circ b - a,\ (a \circ b - a)^{-} \circ a \circ b\right)$$

is such that

$$r_{B^{op}} = r_B^{-1}$$

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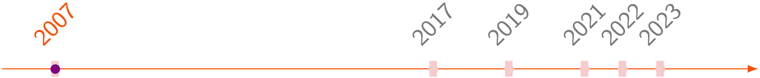
In the last years...



- [Rump]: Braces
- [Guarnieri, Vendramin]: Skew braces
  [Catino, Colazzo, S.]: Semi-braces
- [Jespers, Van Antwerpen]: Semi-braces
- [Catino, Mazzotta, S.]: Inverse semi-braces
  [Catino, Colazzo, S.]: Generalized semi-braces
- [Catino, Mazzotta, Miccoli, S.]: Weak braces
- [Catino, Mazzotta, S.]: Dual weak braces
- [Doikou, Rybołowicz]: Near braces
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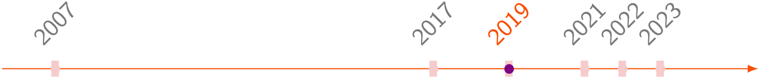

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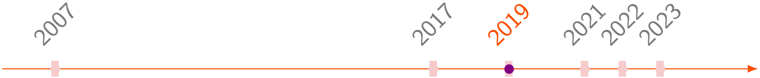

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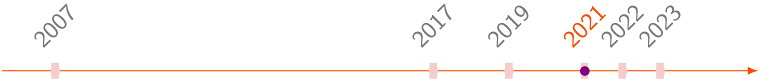

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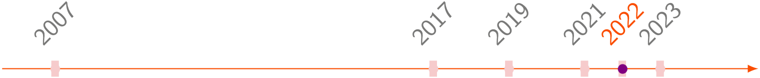

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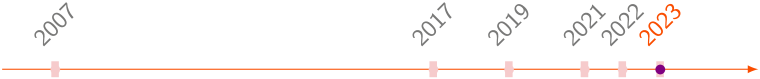

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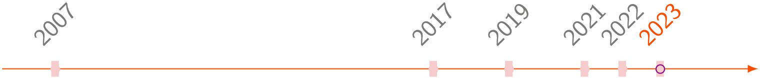

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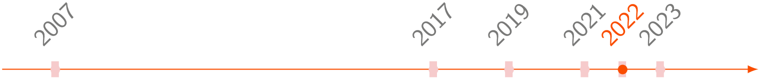

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Inverse semigroups

A semigroup S is called *inverse* if, for each $a \in S$, there exists a unique $a^{-1} \in S$ satisfying

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$

Such an element a^{-1} is called the *inverse of a*. ☞ [Petrich - 1984].

- $\left(a^{-1}\right)^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$, for all $a, b \in S$.
- If $f : S \to T$ is a homomorphism between inverse semigroups, then $f\left(a^{-1}\right) = f(a)^{-1}$, for every $a \in S$.

- The set of idempotent elements $\mathrm{E}(S) = \{a\,a^{-1} \mid a \in S\}$.
- Moreover, idempotents commute each other.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

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Weak braces

Definition (Catino, Mazzotta, Miccoli, S. - 2022)

A triple $(S, +, \circ)$ is said to be a *weak brace* if $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

- $\forall\, a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c$,
- $\forall\, a \in S \qquad a \circ a^{-} = -a + a$,

where $-a$ and a^{-} denote the inverses of $(S, +)$ and $(S, \circ)$.

In any weak brace $\mathrm{E}(S, +) = \mathrm{E}(S, \circ)$ thus we will simply write $\mathrm{E}(S)$. As a consequence, if $|\mathrm{E}(S)| = 1$, then $(S, +, \circ)$ is a skew brace.

- Skew braces are special instances of weak braces.
- If $(S, +)$ is a Clifford semigroup, then $(S, +, +)$ and $(S, +, +^{op})$ are weak braces called the *trivial* and the *almost trivial weak brace* on $(S, +)$, respectively.

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Solutions associated to a weak brace

A key result

If $(S, +, \circ)$ is a weak brace, then

$$\forall\ a \in S, e \in \mathrm{E}(S) \qquad e + a = e \circ a.$$

In particular, $\mathrm{E}(S)$ is a trivial weak brace contained in S.

Theorem (Catino, Mazzotta, Miccoli, S. - 2022)

If $(S, +, \circ)$ is a weak brace, then the map $r_S : S \times S \to S \times S$ given by

$$r_S(a, b) = \left(-a + a \circ b, (-a + a \circ b)^{-} \circ a \circ b\right)$$

is a solution. Moreover, r_S has a behaviour close to bijectivity.

Indeed, we can consider the *opposite weak brace*, i.e., $S^{op} = (S, +^{op}, \circ)$, with $a +^{op} b = b + a$, for all $a, b \in S$, and so we have that

$$r_S\, r_{S^{op}}\, r_s = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Hence, r_s is a completely regular element of $\mathrm{Map}(S \times S)$.

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The additive structure of a weak brace

Theorem

Let $(S,+,\circ)$ be a weak brace. Then, $(S,+)$ is a Clifford semigroup.

Generally, $(S,\circ)$ is not a Clifford semigroup.

Example

Let

- $X := \{1, x, y\}$;
- S the upper semilattice on X with join 1;
- T the commutative inverse monoid on X with identity 1 such that $xx = yy = x$ and $xy = y$;
- $\tau := (xy) \in \mathrm{Aut}(S)$ and $\sigma : T \to \mathrm{Aut}(S)$ defined by $\sigma(1) = \sigma(x) = \mathrm{id}_S$ and $\sigma(y) = \tau$.

Then, considered the trivial weak braces on S and T, $S \rtimes_\sigma T$ is a weak brace such that $(S \times T, \circ)$ is not Clifford since

$$(y,y)\circ(y,y)^- = (y,x) \quad \text{and} \quad (y,y)^- \circ (y,y) = (x,x).$$

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Weak braces coming from RB-operators

Definition (Catino, Mazzotta, S. - 2023)

If $(S,+)$ is a Clifford semigroup, any map $\mathfrak{R}: S \to S$ satisfying

$$\forall\ a,b \in S \qquad \mathfrak{R}(a)+\mathfrak{R}(b)=\mathfrak{R}(a+\mathfrak{R}(a)+b-\mathfrak{R}(a))$$
$$a+\mathfrak{R}(a)-\mathfrak{R}(a)=a$$

is called *Rota–Baxter operator* on $(S,+)$.

Bearing in mind [Guo, Lang, Sheng - 2021] and [Bardakov, Gubarev - 2023] if $\mathfrak{R}$ is an RB-operator on a Clifford semigroup $(S,+)$, the operation defined by

$$a \circ_{\mathfrak{R}} b := a+\mathfrak{R}(a)+b-\mathfrak{R}(a)$$

gives rise to a weak brace $(S,+,\circ_{\mathfrak{R}})$ with $(S,\circ_{\mathfrak{R}})$ a Clifford semigroup.

Instances of Rota–Baxter operators are:

- if $(S,+)$ is a group, maps $\mathfrak{R} \in \mathrm{End}(S,+)$ such that $\mathfrak{R}^2=\mathfrak{R}$ and $\mathrm{Im}\,\mathfrak{R} \subseteq Z(S,+)$.
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Weak braces coming from RB-operators

Definition (Catino, Mazzotta, S. - 2023)

If $(S,+)$ is a Clifford semigroup, any map $\mathfrak{R}: S \to S$ satisfying

$$\forall\ a, b \in S \qquad \mathfrak{R}(a) + \mathfrak{R}(b) = \mathfrak{R}(a + \mathfrak{R}(a) + b - \mathfrak{R}(a))$$
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Dual weak braces

Definition (Catino, Mazzotta, S. - 2023)

A weak brace $(S,+,\circ)$ is called *dual weak brace* if $(S,\circ)$ is a Clifford semigroup.

If $(S,+,\circ)$ is a dual weak brace, then the solution r_S has also a behaviour close to the non-degeneracy in the sense that

$$\lambda_a\lambda_{a^-}\lambda_a = \lambda_a, \qquad \lambda_{a^-}\lambda_a\lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \quad \lambda_a\lambda_{a^-} = \lambda_{a^-}\lambda_a$$

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for every $a \in S$. Hence, λ_a, ρ_a are completely regular elements in $\mathrm{Map}(S)$.

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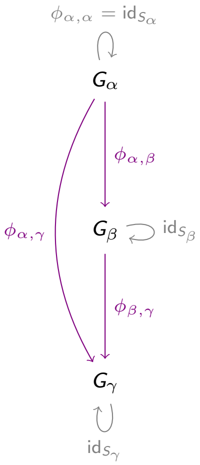
Strong semilattices of groups



☞ [Petrich - 1984]:

- Let Y be a (lower) semilattice.
- Let $\{G_\alpha \mid \alpha \in Y\}$ be a family of disjoint groups.
- For each pair α, β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ be a group homomorphism such that
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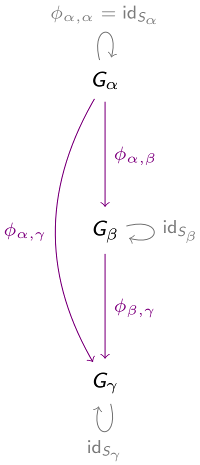

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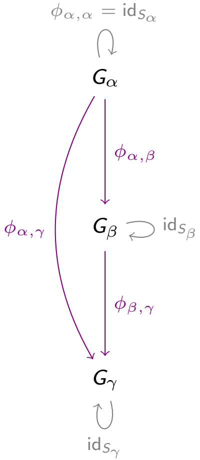

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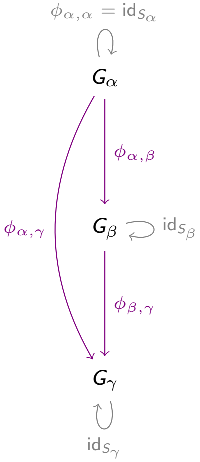

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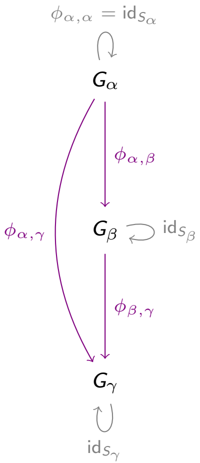

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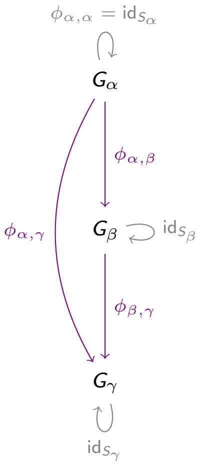

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Strong semilattice of skew braces

Theorem (Catino, Mazzotta, S. - 2023)

- *Let Y be a (lower) semilattice.*
- *Let $\{B_\alpha \mid \alpha \in Y\}$ be a family of disjoint skew braces.*
- *For each $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ be a skew brace homomorphism such that*
  1. $\phi_{\alpha,\alpha} = \mathrm{id}_{B_\alpha}$, *for every $\alpha \in Y$;*
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Then, $S := \bigcup_{\alpha \in Y} B_\alpha$ endowed with

$$a + b := \phi_{\alpha,\alpha\beta}(a) + \phi_{\beta,\alpha\beta}(b)$$
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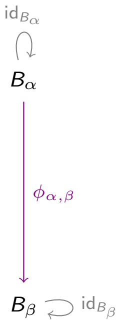
An easy example



Let us consider:

- $Y = \{\alpha, \beta\}$, with $\alpha > \beta$;
- B_α the trivial skew brace on the cyclic group C_3;
- B_β the trivial skew brace on the symmetric group Sym_3;
- $\phi_{\alpha,\beta} : C_3 \to \mathrm{Sym}_3$ the homomorphism given by

$$\phi_{\alpha,\beta}(0) = \mathrm{id}_3,\ \phi_{\alpha,\beta}(1) = (123),\ \phi_{\alpha,\beta}(2) = (132).$$

Then, $S = B_\alpha \bigcup B_\beta$ endowed with the operation given by

$$\forall a \in B_\alpha, b \in B_\beta \quad a + b := \phi_{\alpha,\beta}(a) + \phi_{\beta,\beta}(b)$$
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is a (not trivial) dual weak brace.

Homomorphisms between skew braces

Problem

Finding homomorphism between skew braces for constructing dual weak braces.

This problem already emerged in literature:

- [Cedó - 2018], [Vendramin - 2019] pose the problem of computing the automorphism groups of skew braces of size p^n.
- [Zenouz - 2019] determines the automorphism group of skew braces of order $p > 3$.
- [Puljić, Smoktunowicz, Zenouz - 2022] describe $\mathbb{F}_p$-braces of cardinality p^4 which are not right nilpotent.
- [Rathee, Yadav - 2023] deal with automorphisms of skew braces for developing some general homological and cohomological aspects related to skew braces.
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Strong semilattice of solutions

Theorem (Catino, Colazzo, S. - 2021)

- *Let Y be a (lower) semilattice.*
- *Let $\{r_\alpha \mid \alpha \in Y\}$ be a family of disjoint solutions on X_α indexed by Y.*
- *For each $\alpha \geq \beta$ let $\phi_{\alpha,\beta} : X_\alpha \to X_\beta$ be a map.*
- *Let $X := \bigcup_{\alpha \in Y} X_\alpha$ and $r : X \times X \longrightarrow X \times X$ the map defined as*

$$r(x, y) := r_{\alpha\beta}\left(\phi_{\alpha,\alpha\beta}(x), \phi_{\beta,\alpha\beta}(y)\right),$$

for all $x \in X_\alpha$ and $y \in X_\beta$.

If the following conditions are satisfied:

1. $\phi_{\alpha,\alpha} = \mathrm{id}_{X_\alpha}$, *for every $\alpha \in Y$,*
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3. $(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})\, r_\alpha = r_\beta\, (\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})$, *for all $\alpha, \beta \in Y$ such that $\alpha \geq \beta$,*

then r is a solution on X, called strong semilattice of the solutions r_α.

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The solutions associated to dual weak braces

Theorem

Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace. Then, the solution r associated to S is the strong semilattice of the bijective non-degenerate solutions r_α associated to each skew brace B_α.

Corollary

Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a ***finite*** *dual weak brace and r the solution associated to S. Then, $r^{2k+1} = r$ with $2k = \text{lcm}\{p(r_\alpha) \mid \alpha \in Y\}$.*

As a particular case, the solution r associated to a finite dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ is *cubic*, i.e., $r^3 = r$, if and only if each r_α is involutive, i.e., each B_α is a brace.

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Ideals of dual weak braces

A *normal subsemigroup* I of a Clifford semigroup S is a subset I of S such that

1. $E(S) \subseteq I$;
2. $\forall\, a, b \in I \quad a\, b \in I \quad \text{and} \quad a^{-1} \in I$;
3. $\forall\, a \in S \quad a^{-1}\, I\, a \subseteq I$.

Definition

A subset I of a dual weak brace S is an *ideal* of $(S, +, \circ)$ if

1. I is a normal subsemigroup of $(S, +)$;
2. I is a normal subsemigroup of $(S, \circ)$;
3. $\lambda_a(I) \subseteq I$, for every $a \in S$;

If I is an ideal, the relation $\sim_I$ on S given by

$$\forall\, a, b \in S \quad a \sim_I b \iff a - a = b - b \ \text{ and } \ -a + b \in I,$$

is a congruence of $(S, +, \circ)$.

Ideals of dual weak braces

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Some examples of ideals

Clearly, every ideal is a dual weak sub-brace of a dual weak brace S. Moreover, every ideal I is such that $\mathrm{E}(S) \subseteq I$.

- S and $\mathrm{E}(S)$ are trivial ideals of S.
- The set

  $$\mathrm{Soc}(S) := \{a \mid a \in S, \ \forall\, b \in S \quad a + b = a \circ b \quad \text{and} \quad a + b = b + a\}$$

  is an ideal of S called the *socle* of S.
- Denoted by $\zeta(S, \circ)$ the center of $(S, \circ)$, the set

  $$\mathrm{Ann}(S) := \mathrm{Soc}(S) \cap \zeta(S, \circ),$$

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A characterization of ideals

Theorem (Catino, Mazzotta, S. - 2023)

Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace, I_α an ideal of each skew brace B_α. Set $\psi_{\alpha,\beta} := \phi_{\alpha,\beta}|_{I_\alpha}$, for all $\alpha \geq \beta$, if $\phi_{\alpha,\beta}(I_\alpha) \subseteq I_\beta$, for any $\alpha > \beta$, then

$$I = [Y; I_\alpha; \psi_{\alpha,\beta}]$$

is an ideal of S. Conversely, every ideal of S is of this form.

Remark: $\mathrm{Soc}(S) \subseteq \bigcup_{\alpha \in Y} \mathrm{Soc}(B_\alpha)$ but, in general,

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Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ *be a dual weak brace,* $\psi_{\alpha,\beta} := \phi_{\alpha,\beta}|_{\text{Soc}(B_\alpha)}$*, for all* $\alpha \geq \beta$*, and assume that* $I := [Y; \text{Soc}(B_\alpha); \psi_{\alpha,\beta}]$ *is an ideal of* S*. Then,* $I = \text{Soc}(S)$*.*

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Thank you!

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