Main features of fundamental (graded) algebras

(based on a joint work with A. Giambruno and E. Spinelli)

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Fix an algebraically closed field of characteristic zero, F. Some notations:

- A is an associative algebra,
- $X := \{x_1, x_2, \dots\}$ is a countable set,
- $F\langle X\rangle$ is the free associative algebra over F generated by X.

Definition

An element $f(x_1, \dots, x_n)$ of $F\langle X\rangle$ is a **polynomial identity** (or a *PI*) for A if $f(a_1, \dots, a_n) = 0_A$ for every $a_1, \dots, a_n \in A$.
A is a **PI-algebra** if A satisfies a non-trivial polynomial identity $f \neq 0_{F\langle X\rangle}$.
Let $Id(A) := \{f \mid f \in F\langle X\rangle,\ f \text{ } PI \text{ for } A\}$.

Specht Problem

One of the main questions in PI theory is

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Char F=0, A a PI-algebra $\Rightarrow$ Id(A) is finitely generated as a T ideal?

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A positive solutions was given by Kemer in 1987.

One of the main steps of the proof is the following

Kemer's Representability Theorem

Let A be a *PI*-finitely generated algebra over a field F of characteristic zero. Then there exists a finite-dimensional algebra over a field extension of F which has the same polynomial identities of A.

One of the tools to prove Kemer's Representability Theorem was the introduction of the so called **fundamental algebras**. The main reason lies on one of their properties.

Any finite-dimensional algebra has the same identities as a finite direct sum of fundamental algebras.

Algebra index

Fix a finite-dimensional algebra A.

Wedderburn Malcev Decomposition

$$A = A_{ss} + J(A),$$

where $J(A)$ is the **Jacobson radical** (which is a nilpotent ideal of nilpotency index $s_A + 1$) and where A_{ss} is a **maximal semisimple subalgebra** of A. A_{ss} can be written as a direct sum of simple algebras: $A_{ss} = A_1 \oplus \ldots \oplus A_n$.

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Definition

The t-$s(A) := (dim_F(A_{ss}), s_A)$ is said to be the **algebra index** of A.

Understanding Fundamental Algebras

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Facts

If f is alternating in $d > dim_F(A)$ variables, then $f \in Id(A)$.

Now assume that you have a polynomial f alternating in an arbitrary number of sets each with $d \leq dim_F(A_{ss})$ variables. It might be that there exists a non-zero evaluation in A_{ss}. Now if you allow some of these sets (say s) to be of cardinality $dim_F(A_{ss}) + 1$, you *might* find a non-zero evaluation on A only if you require that $s \leq s_A$, where $s_A + 1$ is the nilpotency index of $J(A)$.
Now, in some sense, a *fundamental algebra* is an “extreme” algebra, in the sense that it realizes the maximal possible number of alternations.

(almost a) Definition

A is **fundamental** if there exists a polynomial $f \notin Id(A)$ which is alternating in an arbitrary number of sets, each of cardinality $\underline{dim_F(A_{ss})}$, and which is also alternating in $\underline{s_A}$ sets of cardinality $\underline{dim_F(A_{ss}) + 1}$.

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We can see this sense of "extreme algebra" also in an other equivalent definition:

An other possible Definition

An algebra A is **fundamental** if it is not PI-equivalent to $C_1 \oplus \cdots \oplus C_r$, where C_i is a finite dimensional algebra with $t\text{-}s(C_i) < t\text{-}s(A)$ for every $1 \leq i \leq r$.

Examples of fundamental algebras

- The **simple** algebra $A \cong M_n(F)$
  $(\prod_{i=1}^{\nu} \sum_{\sigma \in S_{n^2}} (-1)^\sigma y_1^{(i)} x_{\sigma(1)}^{(i)} y_2^{(i)} x_{\sigma(2)}^{(i)} \dots y_{n^2}^{(i)} x_{\sigma(n^2)}^{(i)} y_{n^2+1}^{(i)})$;
- A **nilpotent** algebra with nilpotency index n
  $(x_1 \dots x_{n-1})$;
- The algebra of **upper block triangular matrices** $UT(d_1, \dots, d_n)$, which is the subalgebra of the matrix algebra M_d, where $d := d_1 + \dots + d_n$, consisting of all matrices of the type

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ 0 & A_{22} & \dots & A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{nn} \end{pmatrix}, \ A_{ij} \in M_{d_i \times d_j} \text{ fo every } 1 \leq i \leq j \leq n.$$

Graded polynomial identities

Some notations:

- A is an associative superalgebra (i.e. $A = A^{(0)} \oplus A^{(1)}$ s.t. $A^{(i)}A^{(j)} \subseteq A^{(i+j)}$ for every $i, j \in \mathbb{Z}_2$),
- $Y := \{y_1, y_2, \ldots\}$ and $Z := \{z_1, z_2, \ldots\}$ are (disjoint) countable sets,
- $F\langle Y \cup Z\rangle$ is the free associative algebra over F generated by $Y \cup Z$ ($\rightsquigarrow$ structure of superalgebra: $\deg y_i = 0$, $\deg z_i = 1$ for every $i \geq 1$).

Definition

An element $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ of $F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$**-graded polynomial identity** or a **superidentity** ($f \equiv 0$) for a superalgebra $A = A^{(0)} \oplus A^{(1)}$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0_A$ for every $a_1, \ldots, a_m \in A^{(0)}$ and $b_1, \ldots, b_n \in A^{(1)}$.

Examples

Let us look at some graded polynomial identities:

- $A := \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \subseteq M_3(F)$ with grading defined by $A^{(0)} = \mathrm{span}_F\{e_{13}\}$ and $A^{(1)} = \mathrm{span}_F\{e_{12}, e_{23}\}$.

$e_{12}e_{23} = e_{13} \neq 0_A \Rightarrow z_1 z_2 \not\equiv 0$
But: $y_1 y_2 \equiv 0$.

- $B := M_2(F)$ with grading defined by $B^{(0)} = \mathrm{span}_F\{e_{11}, e_{22}\}$ and $B^{(1)} = \mathrm{span}_F\{e_{12}, e_{21}\}$.

$[y_1, y_2] \equiv 0$.

Fix a finite-dimensional superalgebra A.

Wedderburn Malcev Decomposition

$$A = A_{ss} + J(A),$$

where $J(A)$: Jacobson radical (which is a homogeneous nilpotent ideal) and where A_{ss} is a maximal semisimple subalgebra of A having an induced $\mathbb{Z}_2$-grading. A_{ss} can be written as a direct sum of graded simple algebras: $A_{ss} = A_1 \oplus \ldots \oplus A_n$.

Main definition

Definition

A is **fundamental** if there exists a superpolynomial $f \notin Id_2(A)$ which is alternating in an arbitrary number of couple of sets $\{Y_i, Z_i\}_i$, each of cardinality $(dim_F(A_{ss}^{(0)}), dim_F(A_{ss}^{(1)}))$, and which is also alternating in a even sets of cardinality $dim_F(A_{ss}^{(0)}) + 1$ and in b odd sets of cardinality $dim_F(A_{ss}^{(1)}) + 1$, where $a + b = s_A$, the nilpotency index of $J(A)$.

Recap on Simple Superalgebras

A $\mathbb{Z}_2$-grading on M_m is called **elementary** if there exists an m-tuple $(g_1, \ldots, g_m) \in \mathbb{Z}_2^m$ such that $e_{ij} \in M_m^{(g)}$ if, and only if $g = g_i - g_j$.

Let A be a simple superalgebra. A is (isomorphic to) a superalgebra of the following type

(a) $M_{k,l} := M_{k+l}$ with $k \geq l \geq 0$, $k \neq 0$, endowed with the grading induced by the $(k+l)$-tuple $(\underbrace{0, \ldots, 0}_{k \text{ times}}, \underbrace{1, \ldots, 1}_{l \text{ times}})$;

(b) $M_m + tM_m$, where $t^2 = 1_F$, with grading (M_m, tM_m). It can be realized as the homogeneous subalgebra $\left\{ \begin{pmatrix} C & D \\ D & C \end{pmatrix} \right\}$ of the full matrix algebra M_{2m} with the grading induced by the $(2m)$-tuple $(\underbrace{0, \ldots, 0}_{m \text{ times}}, \underbrace{1, \ldots, 1}_{m \text{ times}})$.

Examples of Fundamental Superalgebras

Proposition

Every finite-dimensional simple superalgebra is fundamental.

Upper block triangular matrix algebras

Let $((A_1, \alpha_1) \dots, (A_n, \alpha_n))$ be a sequence of simple superalgebras, each with grading $\alpha_j := (\alpha_{j,1}, \dots, \alpha_{j,s_j}) \in \mathbb{Z}_2^{s_j}$, where

$$s_j := \begin{cases} k_j + l_j & \text{if } A_j \cong M_{k_j,l_j}, \\ 2n_j & \text{if } A_j \cong M_{n_j} + tM_{n_j} \end{cases},$$

and set $\eta_n = s_1 + \cdots + s_n$.

Upper block triangular matrix algebras

Finally, let us define $UT(A_1, \dots, A_n) \subseteq UT(s_1, \dots, s_n)$ consisting of all matrices of the type

$$\begin{pmatrix} A_{11} & * & \dots & * \\ 0 & A_{22} & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{nn} \end{pmatrix}, \ A_{ii} \in A_i \ \text{ for every } \ i \in [1, n].$$

Upper block triangular matrix algebras

6 New Results

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Define $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^{\eta_n}$ and, for any n-tuple $\tilde{g} := (g_1, \dots, g_n) \in \mathbb{Z}_2^n$,

$$\alpha_{\tilde{g}} := (\alpha_1 \hat{+} g_1, \dots, \alpha_n \hat{+} g_n) =$$

$$(\alpha_{1,1} + g_1, \dots, \alpha_{1,s_1} + g_1, \dots, \alpha_{n,1} + g_n, \dots, \alpha_{n,s_n} + g_n)$$

Denote any such a $\mathbb{Z}_2$-graded algebra (regardless of $\tilde{g}$) by $UT_{\mathbb{Z}_2}(A_1, \dots, A_n)$.

Why are we studying these algebras?

Theorem [O.M. Di Vincenzo, V.R.T. da Silva, E. Spinelli]

A variety of $\mathbb{Z}_2$-graded PI-algebras of finite basic rank is minimal of superexponent $d \geq 2$ if, and only if, it is generated by a $\mathbb{Z}_2$-graded algebra $UT_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ satisfying $\dim_F(A_1 \oplus \ldots \oplus A_n) = d$.

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We have proved:

Theorem [A. Giambruno, E. Spinelli, E.P.]

The superalgebras $UT_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ are fundamental.

Upper block triangular matrix algebras with identification

Let $(A_1, \ldots, A_n)$ be a sequence of simple superalgebras.
For any partition $\mathcal{I}$ of $\{1, 2, \ldots, n\}$ such that

$$\text{if } \ u \in \mathcal{I}, \quad \text{then } \ A_i \cong A_j \ \ \forall i, j \in u,$$

let us denote by $A := UT^{\mathcal{I}}_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ the subalgebra of $B := UT_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ (with grading defined by the map $\alpha_{\tilde{g}}$) obtained from B by identifying all the simple components A_i whose subscript indices i belong to the same element of $\mathcal{I}$.

Example

If $n = 3$ and $\mathcal{I} := \{\{1, 3\}, \{2\}\}$, $UT^{\mathcal{I}}_{\mathbb{Z}_2}(F, F, F) = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & e \\ 0 & 0 & a \end{pmatrix}, \ a, b, c, d \in F \right\}.$

Theorem [A. Giambruno, E. Spinelli, E.P.]

The superalgebra $UT_{\mathbb{Z}_2}^{\mathcal{I}}(A_1, \ldots, A_n)$ is <u>*not* fundamental</u> if, and only if, there exists $\{i, i+1\} \in \mathcal{I}$ such that $A_i \cong M_{k_i,l_i}$ and $g_i = g_{i+1}$.

Thank you for listening!