Continuum braid groups

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Outline

1. Quantum Group
2. Continuum Quantum Group
3. Braid Group
4. Continuum Braid Group

Braid Group and Quantum Groups

Let X_n a Dynkin Diagram of finite or affine type.
Dynkin diagrams of finite or affine type classify semisimple Lie algebras and Affine Lie Algebra, we denote by $U(X_n)$ the universal enveloping algebra associated to the related Lie Algebra.
Let $U_q^{DJ}(X_n)$ be the quantum group introduced by Drinfeld and Jimbo

Braid Group and Quantum Group

DD

$\Updownarrow$

Lie Alg

$\Updownarrow$

U.E.A

$\Updownarrow$

QG.

EXAMPLE

A_n F.D.

sl_n

$\Updownarrow$

$\mathcal{U}(sl_n)$

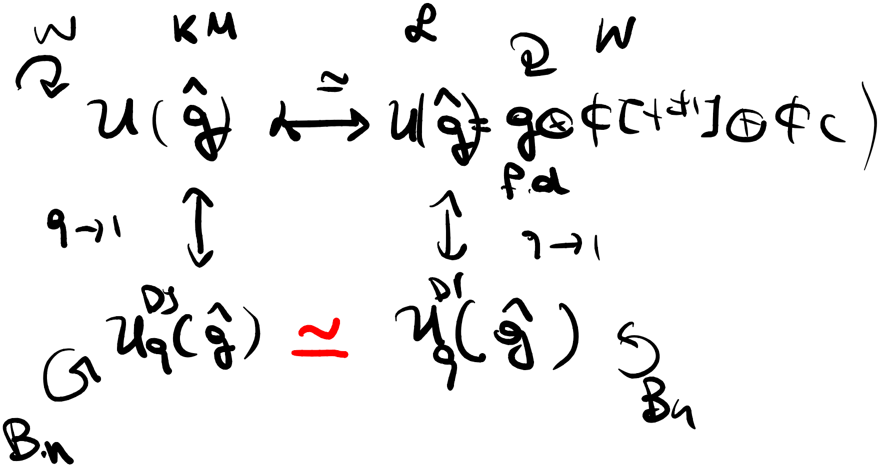
$\Updownarrow$ $\uparrow q \to 1$

$\mathcal{U}_q^{DJ}(sl_n)$

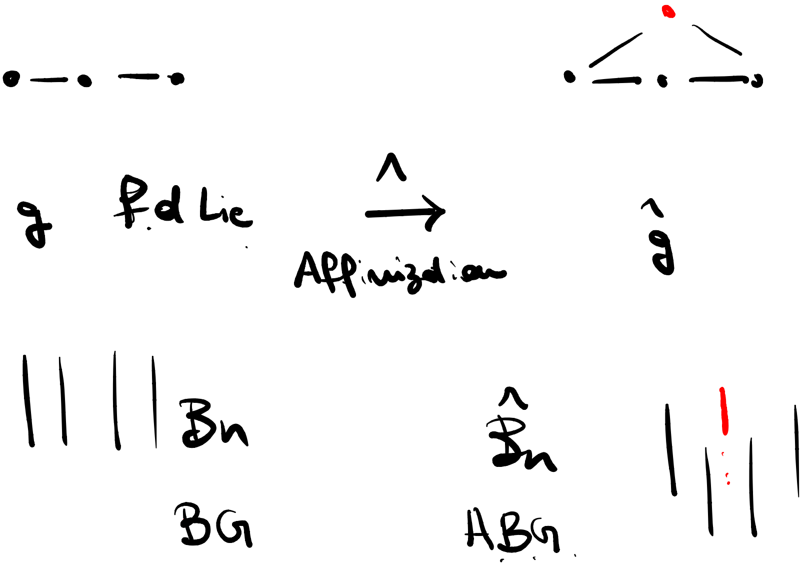
Braid Group and Quantum Group

If X_n is of affine type we will denote by $U_q^{Dr}(X_n)$ its Drinfeld realization. Drinfeld realization is a quantum analog of the loop algebra realization of the affine Kac-Moody Lie algebras.
The isomorphism between the two presentations was defined by Beck and proved to be an isomorphism by Damiani. This is done using the braid group action defined on $U_q^{DJ}(X_n)$ by Lusztig. The role played by the braid group in the case of quantum groups is analogous to the role played by the Weyl group in the case of classical analogs of quantum groups, that is, the universal enveloping algebra $U(X_n)$ of a Lie algebra .

Example



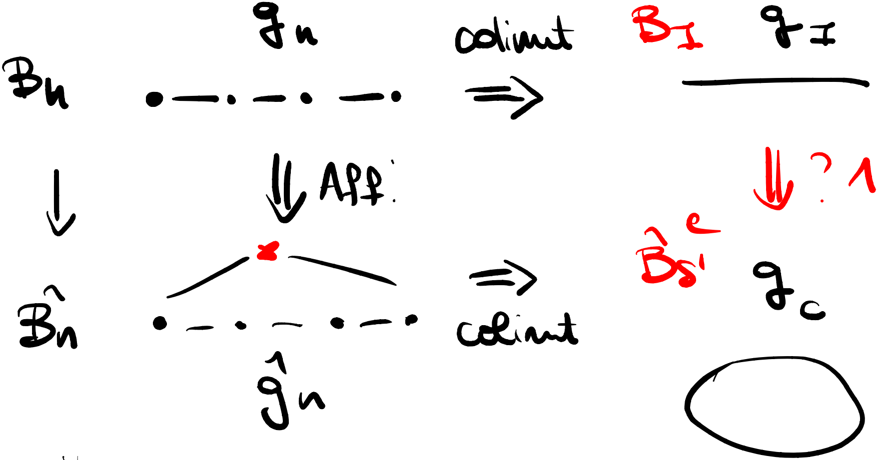
Example



Continuum Kac–Moody algebra

In 2019 Appel, Sala and Schiffman introduced a generalization of Kac Moody algebras, called continuum Kac–Moody algebra.
The defining datum of a continuum Kac–Moody algebra is a continuum analogue of a Dynkin Diagram.
The simplest non–trivial examples of continuum Kac–Moody algebras are the Lie algebras of the interval $I = (0, 1)$ and the circle.
They construct an explicit quantization of these algebras, which they refer to as a continuum quantum group. They shows that the latter is similarly realized as an uncountable colimit of Drinfeld–Jimbo quantum groups.

Example



Is it possible to define an affinization from $\mathfrak{g}_I$ to $\mathfrak{g}_{S^1}$? Does this lift to the quantum curve?

Continuum Kac–Moody algebra

Their construction is closely related to that of usual Kac–Moody algebras. Their Cartan datum encodes the topology of a one–dimensional real space and can be thought of as a generalization of a quiver, where vertices are replaced by connected intervals.

e.g. Roughly, they are generated by infinitely–many $\mathfrak{sl}_2$–triples indexed by pairs of points (or equivalently intervals) in the circle $\mathcal{S}^1$ and commutation relations depending upon their mutual position in $\mathcal{S}^1$.

Continuum Kac-Moody algebra, Example

Ex $\mathfrak{g}_I$

$\{f_\alpha, e_\alpha\}$ $\alpha = (a, b] \subset I = (0, 1)$

+ Relation $e_\alpha, f_\alpha, h_\alpha \simeq \mathfrak{sl}_2$

$\mathrm{ad}^2_{e_\beta}(e_\alpha) = 0$ if $\alpha \to \beta$

⇓

$\alpha = (a, b]$

$\beta = (b, c]$

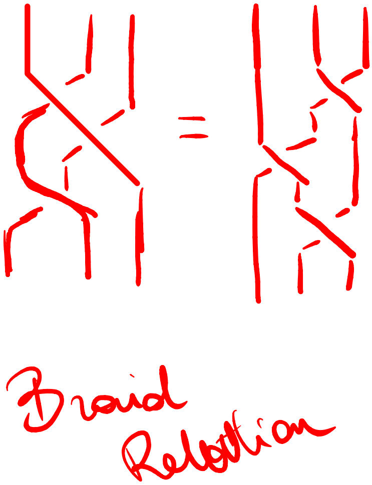
Braid Group

The braid group on n strands B_n has a presentation with $n-1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i-j| = 1; \text{Braid relation} \quad (1)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \neq 1. \quad (2)$$

Braid Group



Extended Affine Braid Group

The extended affine braid group on n strands $\widehat{B}_n^e$ has a presentation with $n+1$ generators $\sigma_1, \sigma_2, ..., \sigma_n, \tau$ subject to the relations:

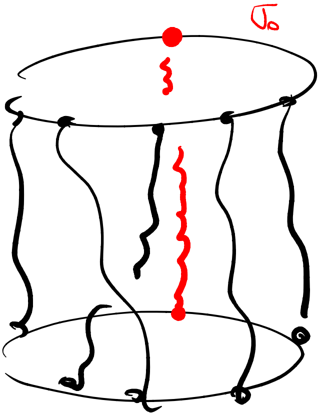
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i-j| = 1; \text{Braid relation} \tag{3}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \neq 1; \tag{4}$$

$$\tau \sigma_i \tau^{-1} = \sigma_{i+1} \tag{5}$$

where the indices have to be understood modulo n.

Extended Affine Braid Group



Birman-Ko-Lee Presentation

[Thorem Birman-Ko-Lee 1998]

B_n has a presentation given by: generators $a_{r,s}$ $1 \leq r < s \leq n$ and relations

$$a_{s,t}a_{q,r} = a_{q,r}a_{s,t} \text{ if } (t-r)(t-q)(s-r)(s-q) > 0; \tag{6}$$

$$a_{s,t}a_{r,s} = a_{r,t}a_{s,t} = a_{r,s}a_{r,t} \text{ if } 1 \leq r < s < t \leq n; \tag{7}$$

The isomorphism

$$a_{r,s} = (\sigma_{s-1}\sigma_{s-2}\ldots\sigma_{r+1})\sigma_r(\sigma_{r+1}^{-1}\ldots\sigma_{s-2}^{-1}\sigma_{s-1}^{-1}) \tag{8}$$

Remark

$a_{r,s}$ is the exchange of the r-th strand with the s-th strand.
Relation (6) asserts that $a_{s,t}$ and $a_{q,r}$ commute if t and s do not separate r and q.
Relation (7) generalizes the braid relation.

BKL Presentation of B_I

Definition

The Braid Group of the interval B_I is the group generated by the elements

$$\sigma_\alpha, \alpha = (a, b] \in I = (0, 1)$$

subject to the relations:

$$\sigma_\beta \sigma_\alpha = \sigma_\alpha \sigma_\beta$$

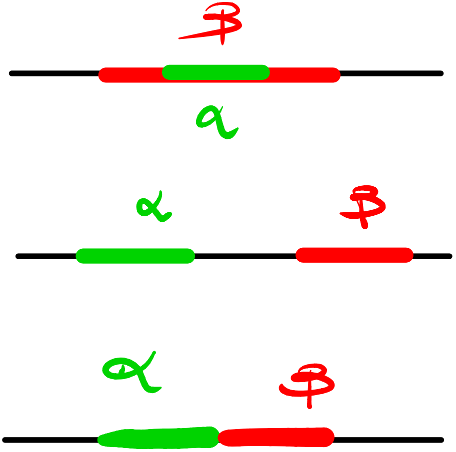
if α and β are one nested in the other or are distant

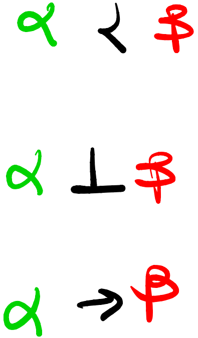
(notation: $\alpha \prec \beta$ or $\beta \prec \alpha$ or $\alpha \perp \beta$)

$$\sigma_\beta \sigma_\alpha = \sigma_{\alpha \cup \beta} \sigma_\beta = \sigma_\alpha \sigma_{\alpha \cup \beta}$$

if β follows α

BKL Presentation of B_I





The Braid Group of the Circle

Definition

The extended Braid Group of the circle $B^e_{S^1}$ is the group generated by the elements

$$\sigma_\alpha, \alpha = (a, b] \subseteq S^1$$

subject to the relations:

$$\sigma_\beta \sigma_\alpha = \sigma_\alpha \sigma_\beta \text{ if } \alpha \prec \beta \text{ or } \beta \prec \alpha \text{ or } \alpha \perp \beta$$

$$\sigma_\beta \sigma_\alpha = \sigma_{\alpha \cup \beta} \sigma_\beta = \sigma_\alpha \sigma_{\alpha \cup \beta} \text{ if } \beta \to \alpha \text{ and } \alpha \cup \beta \neq S^1$$

$$t_x \sigma_\alpha t_x^{-1} = \sigma_{t_x(\alpha)}$$

where $t_x((a, b]) = (a + x, b + x]$

Continumm QG and Continumm BG

Theorem

It is possible to define an analogue of the Drinfeld "new realization" for the Quantum Group of the circle $U_q^{Dr}(S^1)$.
The action of the Extended Braid group of the circle $B^e_{\mathcal{S}^1}$ define an isomorphim between the two presentation.
Loop like generators are found for the algebra which satisfy the relations of Drinfeld's "new realization".

Grazie

GRAZIE!