# On the arithmetic and geometric means of element orders in a finite group

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# Joint work with

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E. Di Domenico, C. Monetta and M. Noce,
*Upper bounds for the product of element orders of finite groups*, J. Algebraic Comb., **57**(2023), 1033–1043.

V. Grazian, C. Monetta and M. Noce,
*On the structure of finite groups determined by the arithmetic and geometric means of element orders*, preprint available at arXiv:2212.13770 [math.GR]

# Functions depending on elements orders

In 2009, Amiri, Jafarian Amiri and Isaacs considered the following function:

$$\psi(G) = \sum_{x \in G} o(x)$$

for any finite group $G$.

H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs,
*Sums of element orders in finite groups*, Comm. Algebra **37** (2009), 2978–2980

The main problem they addressed was to understand to what extent the value of $\psi(G)$ determines properties of the group $G$ itself.

# Some results

Denote by $C_n$ the cyclic group of order $n$.

**Amiri - Jafarian Amiri - Isaacs**

Let $G$ be a group of order $n$. Then $\psi(G) \leq \psi(C_n)$ and $\psi(G) = \psi(C_n)$ if and only if $G$ is cyclic.

H. Amiri and S.M. Jafarian Amiri,
*Sum of element orders on finite groups of the same order*, J. Algebra Appl. **10** (2011), pp. 187–190.

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# An exact upper bound

**Herzog - Longobardi - Maj**

Let $G$ be a non-cyclic group of order $n$. Then $\psi(G) \leq \frac{7}{11}\psi(C_n)$.

M. Herzog, P. Longobardi, M. Maj,
*An exact upper bound for sums of elements order in non-cyclic finite groups*, J. Pure Appl. Algebra **222** (2018), pp. 1628–1642.

The constant:

$\psi(C_2 \times C_2) = 1 \; + \; 2 + \; 2 + \; 2 = 7$

$\psi(C_4) = 1 \; + \; 2 + \; 2 \cdot 4 = 11$

$$\psi(C_{2k} \times C_2) = \frac{7}{11}\psi(C_{4k})$$

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# The product of element orders

In 2013, Tărnăuceanu considered the following function:

$$\rho(G) = \prod_{x \in G} o(x)$$

for any finite group $G$.

M. Tărnăuceanu,
*A note on the product of element orders of finite abelian groups*,
Bull. Malays. Math. Sci. Soc. **36** (2) (2013), 123–1126.

# Upper bound

**Garonzi - Patassini**

Let $G$ be a group of order $n$. Then $\rho(G) \leq \rho(C_n)$ and $\rho(G) = \rho(C_n)$ if and only if $G$ is cyclic.

M. Garonzi and M. Patassini, *Inequalities detecting structural properties of a finite group*, Comm. Algebra **45** (2016), 677–687.

# Natural question

What is the maximal product of element orders among non-cyclic groups of order n?

# Inspiration

$$\rho(C_2 \times C_2) = 1 \cdot 2 \cdot 2 \cdot 2 = 8$$

$$\rho(C_4) = 1 \cdot 2 \cdot 4^2 = 32$$

$$\rho(C_2 \times C_2) = \frac{1}{4}\,\rho(C_4) = \frac{1}{2^2}\,\rho(C_4)$$

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# Conjecture

**Conjecture - Di Domenico - M. - Noce**

Let $G$ be a non-cyclic group of order $n$, and assume that $q$ is the smallest prime dividing $n$. Then

$$\rho(G) \ \leq \ \frac{1}{q^q}\, \rho(C_n).$$

## Some positive results

**Di Domenico - M. - Noce**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime dividing $n$. If **either**

- $G$ admits a Sylow tower, **or**
- $n = p^\alpha q^\beta$, **or**
- $G$ is a Frobenius group,

then

$$\rho(G) \ \leq \ \frac{1}{q^q} \ \rho(C_n)$$

A group $G$ admits a Sylow tower if there exists a normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

such that each $G_{i+1}/G_i$ is isomorphic to a Sylow subgroup of $G$ for every $i \in \{0, \ldots, n-1\}$.

# We can do better...

**Di Domenico - M. - Noce**

Let $G$ be a non-cyclic nilpotent group. Then

$$\rho(G) \le \frac{1}{q^{\frac{n}{q}(q-1)}} \rho(C_n).$$

Notice that

$$\frac{1}{q^{\frac{n}{q}(q-1)}} \le \frac{1}{q^q} \iff q^q \le q^{\frac{n}{q}(q-1)} \iff \frac{q^2}{q-1} \le n$$

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# An example

If $\mathbb{Q}_8$ denotes the quaternion group of order 8, then

$$\rho(\mathbb{Q}_8) = 1 \cdot 2 \cdot 4^6 = 2^{13}$$

$$\rho(C_8) = 1 \cdot 2 \cdot 4^2 \cdot 8^4 = 2^{17}$$

$$\rho(\mathbb{Q}_8) = \frac{1}{2^4}\,\rho(C_8) = \frac{1}{2^{\frac{8}{2}(2-1)}}\,\rho(C_8)$$

# Other interesting function

★ Consider the functions

$$\psi''(G) = \frac{\psi(G)}{|G|^2} \quad \text{and} \quad I(G) = \frac{\rho(G)^{1/|G|}}{|G|}.$$

Notice that

$$|G| \cdot \psi''(G) \quad \text{and} \quad |G| \cdot I(G)$$

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# Advantage

The function $I(G)$ is multiplicative.

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# Recent results

**Theorem (Azad - Khosravi)**

*Let $G$ be a finite group and let $f \in \{\psi'', I\}$.*

(a) *If $f(G) > f(C_2 \times C_2)$, then $G$ is cyclic.*

(b) *If $f(G) > f(Q_8)$, then $G$ is abelian.*

(c) *If $f(G) > f(S_3)$, then $G$ is nilpotent.*

(d) *If $f(G) > f(A_4)$, then $G$ is supersoluble.*

(e) *If $f(G) > f(A_5)$, then $G$ is soluble.*


M. Tărnăuceanu,
*Detecting structural properties of finite groups by the sum of element orders*, Isr. J. Math., **238** (2020), 629–637.

M. B. Azad and B. Khosravi,
*Properties of finite groups determined by the product of their element orders*, Bull. Aust. Math. Soc. 103 (2021), no. 1, 88–95.

## Our question

What about the $p$-nilpotency of a finite group?

A finite group $G$ is said to be $p$-nilpotent when all its elements of $p'$-order determine a subgroup.

# 2-nilpotency

The smallest non-2-nilpotent finite group is the alternating group of degree 4, namely, $A_4$.

Observing that any supersoluble group is 2-nilpotent, one easily get

$$I(G) > I(A_4) \implies G \text{ is supersoluble} \implies G \text{ is 2-nilpotent}$$

# $p$-nilpotency, $p$ odd

Denote by $D_{2n}$ the dihedral group of order $2n$.

**Theorem A [Grazian - M. - Noce]**

Let $G$ be a finite group and let $p$ be an odd prime dividing the order of $G$.

If $I(G) \geq I(D_{2p})$, then either

- $I(G) = I(D_{2p})$ and $G \cong D_{2p}$ or
- $I(G) > I(D_{2p})$ and $G = O_p(G) \times O_{p'}(G)$ with $O_p(G)$ cyclic.

In particular, if $I(G) > I(D_{2p})$, then $G$ is $p$-nilpotent.

Where we denote by $O_p(G)$ and $O_{p'}(G)$ the largest normal $p$-subgroup and $p'$-subgroup of $G$, respectively.

# Remark 1

We point out that if $n$ is an odd integer

$$I(G) = I(D_{2n}) \not\Rightarrow G \simeq D_{2n}$$

**For instance**, if $n = 9$ we have $I(S_3 \times C_3) = I(D_{18})$.

# Remark 2

**Grazian - M. - Noce**

Let $G$ be a finite group whose order is divisible by the odd prime $p$, and suppose $l(G) > l(D_{2p})$.

1. If $p = 3$ then $G$ is cyclic.
2. If $p \leq 5$ then $G$ is nilpotent.
3. If $p \leq 13$ then $G$ is supersoluble.

**N.B.** The choice of primes in this result is sharp

- if $p > 3$ then $l(C_p \times Q_8) > l(D_{2p})$ and $C_p \times Q_8$ is not cyclic
- if $p > 5$ then $l(C_p \times S_3) > l(D_{2p})$ and $C_p \times S_3$ is not nilpotent
- if $p > 13$ then $l(C_p \times A_4) > l(D_{2p})$ and $C_p \times A_4$ is not supersoluble

# A consequence of Theorem A

## Corollary

*Let $G$ be a finite group of odd order and let $p$* **be the smallest prime divisor of** $|G|$.

(a) *If $I(G) > I(D_{2p})$ then $G$ is cyclic.*

(b) *If the number of distinct primes dividing $|G|$ is at most $\frac{p+1}{2}$, then $G$ is cyclic if and only if $I(G) > I(D_{2p})$.*

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The condition

the number of distinct primes dividing $|G|$ is at most $\frac{p+1}{2}$

seems necessary.

Indeed the cyclic group of order $315 = 3^2 * 5 * 7$ satisfies

- smallest prime 3;
- number of distinct primes is 3 with $3 > \dfrac{3+1}{2}$;
- $I(C_{315}) < I(D_{2\cdot 3})$.

Thanks!