Vectorial functions for symmetric cryptography: immersion and insights

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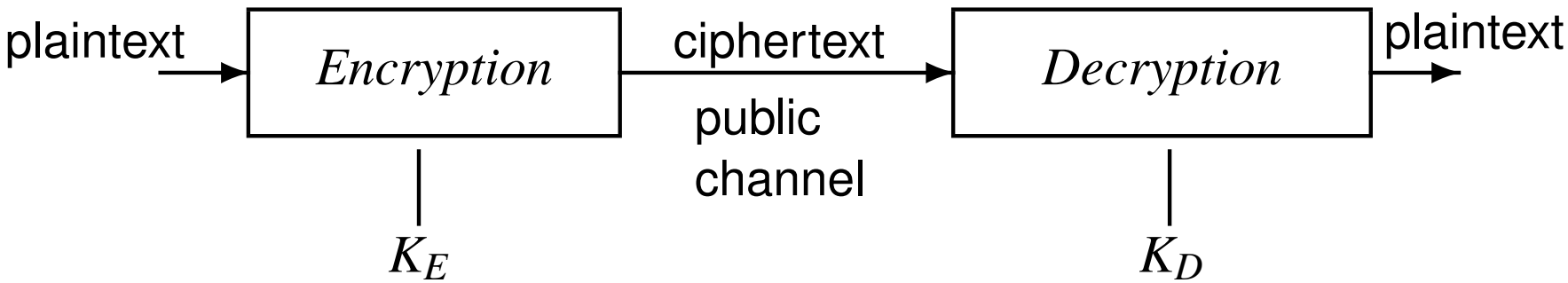
Young Researchers Algebra Conference 2023
July 27, 2023, L'Aquila, Italy

Outline

- Preliminaries on vectorial functions
- Main mathematical problems in symmetric cryptography in block cipher
- the main problem in solving equations over finite fields
  - On the famous equation $x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$
    - Ingredients for the resolution
    - The two related problems for solving $x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$ with $\gcd(n, k) = 1$
    - Solving the two problems
  - Impacts (results obtained in 2021, 2022 and 2023)
    - On the APN-ness property of Kasami functions
    - On the bijectivity property of an "exceptional" family of quadrinomials
    - On the resolution of a new equation $F(x + 1) + F(x) = b$ where F is a power "cryptographic" function

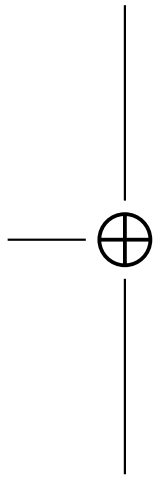
Cryptography



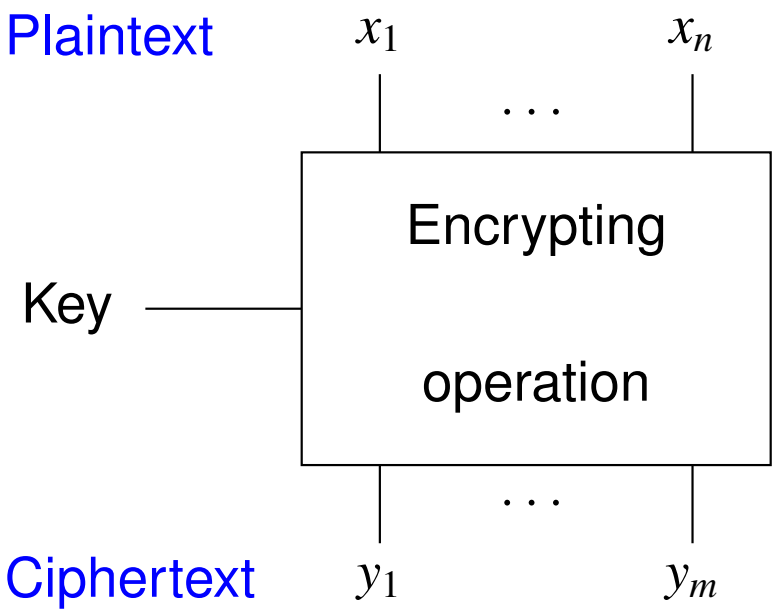
Boolean functions : cryptographic framework

Stream ciphers

Pseudo-random generator with a Boolean function



Block ciphers (AES, DES, etc)



outputs of Boolean functions (depending on the key) over $x_1, \cdots, x_n$

Notation :

- $\mathbb{F}_{2^n}$ the finite field of order 2^n.
- The *absolute trace* over $\mathbb{F}_2$ of an element $x \in \mathbb{F}_{2^n}$ equals $Tr_{2^n/2}(x) = \sum_{i=0}^{n-1} x^{2^i}$.

Symmetric cryptography

Let

$$F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^m}$$

- The (Boolean) functions : $x \mapsto b \cdot F(x) := Tr_{2^m/2}(bF(x))$, $b \in \mathbb{F}_{2^m}$ where $b \neq 0$ are called the *components* of F
  where $Tr_{2^n/2}(x) := \sum_{i=0}^{n-1} x^{2^i}$ (trace function).
- The discrete Hadamard Walsh (Fourier) transform of F :
  $W_F(a,b) := \sum_{x \in \mathbb{F}_{2^n}} (-1)^{Tr_{2^m/2}(bF(x)) - Tr_{2^n/2}(ax)}$
  where $(a,b) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^m} \setminus \{0\}$
- three main important cryptographic parameters for F
  1. Nonlinearity and higher-order nonlinearity
  2. Differential uniformity
  3. Boomerang uniformity

  ☞ To make the cryptanalysis very difficult to implement, we must pay attention when choosing the vectorial function F, which has to follow. several recommendations : **cryptographic criteria** !

☞ Linear Cryptanalysis [Matsui (1993)] $\Rightarrow$ nonlinearity $\mathrm{Nl}(F)$ de $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$,

$$\mathrm{Nl}(F) = \min_{b \in \mathbb{F}_{2^m}, b \neq 0} \{\mathrm{nl}(Tr_{2^m/2}(bF))\} = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n, b \in \mathbb{F}_2^m, b \neq 0} |W_F(a,b)|.$$

★ The higher the value of $\mathrm{Nl}(F)$, the better resistance to linear cryptanalysis.

★ When $n = m$ odd, $\mathrm{Nl}(F)$ is bounded by $2^{n-1} - 2^{\frac{n-1}{2}}$. The functions reaching this upper bound are the AB functions.

★ When $m = 1$, $\mathrm{Nl}(F)$ is bounded by $2^{n-1} - 2^{\frac{n}{2}-1}$. The functions reaching this upper bound are the bent functions (n even).

↪ A very difficult parameter to study mathematically ! !

☞ Differential cryptanalysis [Biham, Shamir (1991)] $\Rightarrow$

- Difference Distribution Table

$$\mathrm{DDT}_F(a,b) = \#\{x \in \mathbb{F}_{2^n},\ F(x+a) + F(x) = b\}, \qquad (1)$$

- Differential uniformity [Nyberg (1992-1993)]

$$\delta_F = \max_{a,b \in \mathbb{F}_{2^n}, a \neq 0} \mathrm{DDT}_F(a,b). \qquad (2)$$

★ The smaller the δ_F value, the better the resistance to differential cryptanalysis.

★ For any function F with $m \leq n$, $\delta_F \geq p^{n-m}$ and $\delta_F \geq 2$ if $p = 2$.

★ If $\delta_F = 1$, the function F (p odd) is non-perfect linear (PN).

★ If $\delta_F = 2$, the function F ($p = 2$) is almost perfectly nonlinear (APN).

A variation of differential cryptanalysis : boomerang cryptanalysis

☞ Boomerang cryptanalysis [Wagner (1999)] $\Rightarrow$

- Boomerang Connectivity Table [Cid. al. (2018)]

$$\mathrm{BCT}_F(a,b) = \#\{x \in \mathbb{F}_{2^n},\ F^{-1}(F(x)+b)+F^{-1}(F(x+a)+b) = a\}. \tag{3}$$

- Boomerang uniformity ([Bourra, Canteaut (2018)])

$$\beta_F = \max_{a,b\in\mathbb{F}_{2^n},a\neq 0} \mathrm{BCT}_F(a,b), \tag{4}$$

★ The smaller the value of β_F, the better the boomerang cryptanalysis resistance (here $n = m$ and F bijective).

★ To secure a block cipher against cryptanalysis boomerang, it must use a function with a value of β_F small.

↪ You have to know how to solve equations over finite fields or at least count the number of solutions : mathematically difficult !

An important example : solving the equation $P_a(x) := x^{2^k+1} + x + a = 0$ over $\mathbb{F}_{2^n}$

- ☞ the polynomial $P_a(x)$ ($a \in \mathbb{F}_{2^n}^\star$) appears in this cryptographic context (APN functions [Budaghyan, Carlet 2006], [Bracken, Tan, Tan 2014], [Canteaut, Perrin, Tian 2019])

but not only since there is an interest in other contexts :

- the general theory of finite fields
- the inverse Galois problem [Abhyankar, Cohen, Zieve, 2000] ;
- the construction of difference sets with Singer parameters [Dillon 2002] ;
- finding cross-correlation between m-sequences [Helleseth, Kholosha, Ness, 2007] ;
- constructing error correcting codes [Bracken, Helleseth 2009] ;
- constructions designs [Tang 2019] ;
- etc.

Solving $x^{2^k+1} + x + a = 0$ over $\mathbb{F}_{2^n}$

What do we know ? (before 2020)

- Solution for $k = 1$ [Berlekamp-Rumsey-Solomon 1967-Williams, 1975]
- Partial results
  Let N_a be the number of solutions of the equation $P_a(x) := x^{2^k+1} + x + a = 0$ in $\mathbb{F}_{2^n}$.
  - In 2004 : [Bluher, 2004] the number of solutions N_a are only 0, 1 and 3 when $\gcd(k, n) = 1$.
  - In 2008 : [Helleseth-Kholosha 2008] got criteria for $N_a = 1$ and an explicit expression of the unique solution when $\gcd(k, n) = 1$.
  - In 2014 : [Bracken-Tan-Tan 2014] presented a criterion for $N_a = 0$ when n is even and $\gcd(k, n) = 1$.

After 2020 : [Kim and SM, FFA 2020] solving completely $P_a(x) = 0$ (but further in 2021 for any n, k and characteristic p) [papers published in the FFA journal with Kim and some members of his research team]

Main ingredients for the resolution of $P_a(x) := x^{2^k+1} + x + a = 0$ over $\mathbb{F}_{2^n}$

Ingredient 1 : Dickson polynomials

- **Definition** The Dickson polynomial of the first kind of degree k in indeterminate x and with parameter $a \in \mathbb{F}_{2^n}^*$ is

$$D_k(x,a) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k}{k-i} \binom{k-i}{i} a^k x^{k-2i},$$

  where $\lfloor k/2 \rfloor$ denotes the largest integer less than or equal to $k/2$. In this talk, we consider only Dickson polynomials $D_k(x,1)$, that we shall denote $D_k(x)$.

- **A lot of properties** For all $x \in \mathbb{F}_{2^n}$, for any integer $h, k > 0$.
  - $\deg D_k = k$;
  - $D_k\left(x + \frac{1}{x}\right) = x^k + \frac{1}{x^k}$;
  - D_k permutation over $\mathbb{F}_{2^n}$ if and only if $\gcd(k, 2^{2n}-1) = 1$;
  - $D_{hk}(x) = D_h(D_k(x))$;
  - etc.

Main ingredients for the resolution of $P_a(x) := x^{2^k+1} + x + a = 0$ over $\mathbb{F}_{2^n}$

Let k and n be two positive integers.
Ingredient 2 : Müller-Cohen-Matthews polynomials $f_{k,d}$ are defined over $\mathbb{F}_{2^n}$ as follows :

$$f_{k,d}(X) := \frac{T_k(X^c)^d}{X^{2^k}}$$

where

$$T_k(X) := \sum_{i=0}^{k-1} X^{2^i} \quad \text{and} \quad cd = 2^k + 1.$$

A basic property for such polynomials $f_{k,2^k+1}$ (when $\gcd(k,n) = 1$) :

1. If k is odd, then $f_{k,2^k+1}$ is a permutation on $\mathbb{F}_{2^n}$ ([Müller, Cohen, Matthews 1994]).
2. If k is even, then $f_{k,2^k+1}$ is 2-to-1 on $\mathbb{F}_{2^n}$ ([Dillon, Dobbertin 2004]).

The two related problems for solving $P_a(x) := x^{q+1} + x + a = 0$; $q = 2^k$, $\gcd(n,k) = 1$

If k is odd, since $\gcd(q-1, 2^n - 1) = 1$, the zeros of $P_a(x)$ are the images of the zeros of $P_a(x^{q-1})$ by the map $x \mapsto x^{q-1}$.

Now $f_{k,q+1}$ is a permutation polynomial of $\mathbb{F}_{2^n}$ by **Ingredient 2**. Therefore, for any $a \in \mathbb{F}_{2^n}^*$, there exists a unique Y in $\mathbb{F}_{2^n}^*$ such that $a = \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)^{\frac{2}{q}}}$. Hence, we have

$$P_a\left(x^{q-1}\right) = x^{q^2-1} + x^{q-1} + \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)^{\frac{2}{q}}} \tag{5}$$

Substituting tx to X in the above identity with $t^{q^2-q} = Y^q T_k\left(\frac{1}{Y}\right)^2$, we get :

$$P_a\left(x^{q-1}\right) = \frac{1}{Y^{q-1}\left(f_{k,q+1}\left(\frac{1}{Y}\right)\right)^{\frac{2}{q}}}\left(X^{q^2-1} + \left(\sum_{i=1}^{k} Y^{q-2^i}\right) X^{q-1} + Y^{q-1}\right)$$

The two related problems for solving $P_a(x) := x^{q+1} + x + a = 0$; $q = 2^k$, $\gcd(n, k) = 1$

we use a key polynomial identity involving Dickson polynomials : a key result due to [Bluher, 2016] :
In the polynomial ring $\mathbb{F}_q[X, Y]$ (where $q := 2^k$) we have the identity

$$X^{q^2-1} + \left(\sum_{i=1}^{k} Y^{q-2^i} \right) X^{q-1} + Y^{q-1} = \prod_{w \in \mathbb{F}_q^*} \left(D_{q+1}(wX) - Y \right).$$

thus

$$P_a\left(x^{q-1}\right) = \frac{1}{Y^{q-1}\left(f_{k,q+1}\left(\frac{1}{Y}\right)\right)^{\frac{2}{q}}} \left(\prod_{w \in \mathbb{F}_q^*} \left(D_{q+1}(wX) - Y \right) \right).$$

☞ when k is odd, finding the zeros of $P_a(x^{q-1})$ amounts to determine preimages of Y under the Dickson polynomial D_{q+1}.

The two related problems for solving $P_a(x) := x^{q+1} + x + a = 0$; $q = 2^k$, $\gcd(n,k) = 1$

When k is even, $f_{k,q+1}$ is 2-to-1, fortunately, we can go back to the odd case by rewriting the equation. Indeed, for $x \in \mathbb{F}_{2^n}$,

$$P_a(x) = x^{2^k+1} + x + a = \left(x^{2^{n-k}+1} + x^{2^{n-k}} + a^{2^{n-k}}\right)^{2^k}$$
$$= \left((x+1)^{2^{n-k}+1} + (x+1) + a^{2^{n-k}}\right)^{2^k}$$

and so

$$\{x \in \mathbb{F}_{2^n} \mid P_a(x) = 0\} = \left\{x+1 \mid x^{2^{n-k}+1} + x + a^{2^{n-k}} = 0,\ x \in \mathbb{F}_{2^n}\right\}. \quad (6)$$

☞ If k is even, then $n-k$ is odd, and we can reduce it to the odd case.

The two related problems for solving $P_a(x) := x^{q+1} + x + a = 0$; $q = 2^k$, $\gcd(n,k) = 1$

To summarize all the above discussions :
Let k and n be two positive integers such that $\gcd(k,n) = 1$.

1. Let k be odd and $q = 2^k$. Let $Y \in \mathbb{F}_{2^n}^*$ be (uniquely) defined by $a = \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)^{\frac{2}{q}}}$. Then,

$$\{x \in \mathbb{F}_{2^n} \mid P_a(x) = 0\} = \left\{ \frac{z^{q-1}}{YT_k\left(\frac{1}{Y}\right)^{\frac{2}{q}}} \mid D_{q+1}(z) = Y,\ z \in \mathbb{F}_{2^n} \right\}.$$

2. Let k be even and $q' = 2^{n-k}$. Let $Y' \in \mathbb{F}_{2^n}^*$ be (uniquely) defined by $a^{q'} = \frac{1}{f_{n-k,q'+1}\left(\frac{1}{Y'}\right)^{\frac{2}{q'}}}$. Then,

$$\{x \in \mathbb{F}_{2^n} \mid P_a(x) = 0\} = \left\{ 1 + \frac{z^{q'-1}}{Y'T_{n-k}\left(\frac{1}{Y'}\right)^{\frac{2}{q'}}} \mid D_{q'+1}(z) = Y',\ z \in \mathbb{F}_{2^n} \right\}.$$

The two related problems for solving $P_a(x) := x^{q+1} + x + a = 0$; $q = 2^k$, $\gcd(n,k) = 1$

☞ we can split the problem of finding the zeros in $\mathbb{F}_{2^n}$ of P_a into two independent problems : with odd k.

A For $a \in \mathbb{F}_{2^n}^*$, find the unique element Y in $\mathbb{F}_{2^n}^*$ such that

$$a^{\frac{q}{2}} = \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)}. \tag{7}$$

B For $Y \in \mathbb{F}_{2^n}^*$, find the preimages in $\mathbb{F}_{2^n}$ of Y under the Dickson polynomial D_{q+1}, that is, find the elements of the set

$$D_{q+1}^{-1}(Y) = \{z \in \mathbb{F}_{2^n}^\star \mid D_{q+1}(z) = Y\}. \tag{8}$$

On Problem A : find Y such that $a^{\frac{q}{2}} = \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)}$

Solve Problem A

- We used
  1. Müller-Cohen-Matthews polynomials
  2. machinery for computing the computational inverse of a family of polynomials
- Problem A amounts to finding the solutions to a linear equation of the form $x^q + x = b$.

An approach to compute the computational inverse of an element of a family of polynomials

For any $x \in \mathbb{F}_{2^n}$, define

$$Q'_{k,k'}(x) = \frac{x^{q+1}}{\sum_{i=1}^{k'} x^{q^i}} \tag{9}$$

where $q := 2^k$, $k' < n$ is the inverse of k modulo n.
We have

- if $\gcd(n,k) = 1$ and k' is odd, then $Q'_{k,k'}$ is a permutation on $\mathbb{F}_{2^n}$ ([Dillon, Dobbertin, 1999]) ;
- a relation ([Dillon, 1999]) :

$$\Delta_k(X) = Q'_{k,k'}\left(X + X^{2^k}\right) = f_{k,q+1}(X + X^2). \tag{10}$$

where $\Delta_k(X) = (X+1)^{2^{2k}-2^k+1} + X^{2^{2k}-2^k+1} + 1$;

An approach to compute the computational inverse of an element of a family of polynomials [continued]

- the polynomial representation of the inverse $R_{k,k'}$ of $Q'_{k,k'}$ on $\mathbb{F}_{2^n}$ has been studied in [Dillon, Dobbertin, 2004] by introducing the following sequences of polynomials :

$$A_1(x) = x,\ A_2(x) = x^{q+1},\ A_{i+2}(x) = x^{q^{i+1}} A_{i+1}(x) + x^{q^{i+1}-q^i} A_i(x), \quad i \geq 1,$$

$$B_1(x) = 0,\ B_2(x) = x^{q-1},\ B_{i+2}(x) = x^{q^{i+1}} B_{i+1}(x) + x^{q^{i+1}-q^i} B_i(x), \quad i \geq 1.$$

The polynomial expression of $R_{k,k'}$ is then
$R_{k,k'}(x) = \sum_{i=1}^{k'} A_i(x) + B_{k'}(x)$.

On Problem A : find Y such that $a^{\frac{q}{2}} = \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)}$

Solve Problem A :

- we use the fact : $\Delta_k(X) = Q'_{k,k'}\left(X + X^{2^k}\right) = f_{k,q+1}(X + X^2)$;
- we use a well-known key decomposition : every element z of $\mathbb{F}_{2^n}^\star$ can be written (twice) $z = c + \frac{1}{c}$ where $c \in \mathbb{F}_{2^n}^\star \cup M$ with $c \neq 1$ and where $M = \{\zeta \in \mathbb{F}_{2^{2n}} \mid \zeta^{2^n+1} = 1\}$
- One has $Y = T + \frac{1}{T}$ where $T \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$ or $T \in M \setminus \{1\}$ where $M = \left\{\zeta \in \mathbb{F}_{2^{2n}} \mid \zeta^{2^n+1} = 1\right\}$ (observe that $M \setminus \{1\} \subset \mathbb{F}_{2^{2n}} \setminus \mathbb{F}_{2^n}$). Consequently :

$$\frac{1}{Y} = \left(\frac{1}{T+1}\right)^2 + \frac{1}{T+1}.$$

On Problem A : find Y such that $a^{\frac{q}{2}} = \frac{1}{f_{k,q+1}\left(\frac{1}{Y}\right)}$

Solve Problem A : We get

$$\begin{aligned} a^{\frac{q}{2}} = \left(f_{k,q+1}\left(\frac{1}{Y}\right)\right)^{-1} &\iff a^{-\frac{q}{2}} = \Delta_k\left(\frac{1}{T+1}\right) \\ &\iff a^{-\frac{q}{2}} = Q'_{k,k'}\left(\left(\frac{1}{T+1}\right)^q + \left(\frac{1}{T+1}\right)\right) \end{aligned} \tag{11}$$

Next, Problem A amounts to finding the solutions to a linear equation of the form $x^q + x = b$, that we solved.

On Problem B : find $D_{q+1}^{-1}(Y) = \{z \in \mathbb{F}_{2^n}^\star \mid D_{q+1}(z) = Y\}$

Solve Problem B :

1. Write $z = c + \frac{1}{c}$ where $c \in \mathbb{F}_{2^n}^\star$ or $c \in M \setminus \{1\}$, where $M = \{\zeta \in \mathbb{F}_{2^{2n}} \mid \zeta^{2^n+1} = 1\}$;
   One gets (using **properties of Dickson polynomials**)
   $Y = D_{q+1}(z) = c^{q+1} + \frac{1}{c^{q+1}} = T + \frac{1}{T}$ with $T = c^{q+1}$.
   The equation $T + \frac{1}{T} = Y$ has two solutions in $\mathbb{F}_{2^n}^\star \cup M$ for any $Y \in \mathbb{F}_{2^n}^\star$ because it is equivalent to the quadratic equation $\left(\frac{T}{Y}\right)^2 + \frac{T}{Y} = \frac{1}{Y^2}$ and that $Tr_{2^{2n}/2}\left(\frac{1}{Y}\right) = 0$.
2. We handle the two situations separately of those two cases that occur depending on the value of $Tr_{2^n/2}\left(\frac{1}{Y}\right)$:
   - If $Tr_{2^n/2}\left(\frac{1}{Y}\right) = 0$, $T + \frac{1}{T} = Y$ has two solutions in $\mathbb{F}_{2^n} \setminus \mathbb{F}_2$;
   - If $Tr_{2^n/2}\left(\frac{1}{Y}\right) = 1$, $T + \frac{1}{T} = Y$ has two solutions in $M \setminus \{1\}$.

Impacts of the resolution of our main equation

- (1) A short-proof APN-ness property of Kasami functions (in 2021)
- (2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials discovered in 2019 from [Perrin, Udovenko, Biryukov, Crypto'2016] (in 2022)
- (3) Solving the equation $X^{2^{3n}+2^{2n}+2^n-1} + (X+1)^{2^{3n}+2^{2n}+2^n-1} = b$ in $\mathbb{F}_{2^{4n}}$ and an alternative proof of a conjecture on the differential spectrum of the related monomial functions (in 2023)

(1) A direct proof of the APN-ness property of Kasami functions

Kasami (APN) function : $F(X) = X^{q^2-q+1}$, $q = 2^k$, $\gcd(k, n) = 1$

Recall that F is said to be *almost perfect nonlinear* (APN) if for every $a \in \mathbb{F}_{2^n}{}^*$ and every $b \in \mathbb{F}_{2^n}$, the equation $F(x) + F(x + a) = b$ has 0 or 2 solutions.

A power function $F(X) = X^d$ is APN if and only if, for every $b \in \mathbb{F}_{2^n}$ the system

$$\begin{cases} X + Y & = 1 \\ X^d + Y^d & = b \end{cases} \tag{12}$$

has at most one pair $\{X, Y\}$ of solutions in $\mathbb{F}_{2^n}$

(1) A direct proof of the APN-ness property of Kasami functions

Let n odd, $F = G_2 \circ G_1^{-1}$ where $G_1(X) = X^{q+1}$ and $G_2(x) = X^{q^3+1}$.

Equation (12) is equivalent to

$$\begin{cases} x^{q+1} + y^{q+1} & = 1 \\ x^{q^3+1} + y^{q^3+1} & = b \end{cases} \tag{13}$$

Theorem : System (13) has at most one pair $\{x, y\}$ of solutions proving that F is APN ([Carlet, Kim, SM, DCC 2021])

Sketch of the proof : Letting $y = x + z$, $v = z^{q^2-1}$ and $c = b + 1$,

- Proving Equation (13) has at most one pair $\{x, y\}$ of solutions is equivalent to proving that $(\star\star)$ has at most one solution v when $(\star)$ has solutions :

$$\begin{cases} \left(\frac{x}{z}\right)^q + \left(\frac{x}{z}\right) & = \frac{1}{v^{\frac{1}{q-1}}} + 1 \quad (\star) \\ (v+1)^{q+1} + cv & = 0 \quad (\star\star) \end{cases}$$

  for every $c \in \mathbb{F}_{2^n}$.

- For every $c \in \mathbb{F}_{2^n}$, the cubic equation $(v+1)^{q+1} + cv = 0$ has at most one solution v in $\mathbb{F}_{2^n}$ such that $Tr_1^n\left(\frac{1}{v^{\frac{1}{q-1}}} + 1\right) = 0$.

(2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials

In Crypto'2016, [Perrin, Udovenko, Biryukov, 2016] discovered the butterfly structure (that contains the Dillon APN permutation of six variables), later generalized [Canteaut, Perrin, Tian, 2019], which turns out to be a powerful approach that generates infinite families of cryptographic functions with best-known nonlinearity and differential properties.

Let m odd, k odd, $\gcd(m,k)=1$, $Q=2^m$, $q=2^k$

$$f_{\underline{\epsilon}}(X) := \epsilon_1 \overline{X}^{q+1} + \epsilon_2 \overline{X}^q X + \epsilon_3 \overline{X} X^q + \epsilon_4 X^{q+1}, \quad \overline{X} = X^Q \tag{14}$$

where

$$\underline{\epsilon} = (\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4) = \begin{cases} (\varepsilon_3,\varepsilon_4,\varepsilon_1,\varepsilon_2), & \text{if } k \text{ is odd} \\ (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4), & \text{if } k \text{ is even,} \end{cases} \in \mathbb{F}_Q^4 \tag{15}$$

and ($\alpha,\beta \in \mathbb{F}_Q$.)

$$\begin{cases} \varepsilon_1 = \alpha^q + \alpha + 1 \\ \varepsilon_2 = \alpha^{q+1} + \alpha + \beta + 1 \\ \varepsilon_3 = \alpha^{q+1} + \alpha^q + \beta + 1 \\ \varepsilon_4 = \alpha^{q+1} + \alpha^q + \alpha + \beta \end{cases} \tag{16}$$

(2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials

[Li, Li, Helleseth, Qu, 2021]

$\varphi_1 = \epsilon_1\epsilon_3 + \epsilon_2\epsilon_4$, $\varphi_2 = \epsilon_1\epsilon_2 + \epsilon_3\epsilon_4$, $\varphi_3 = \epsilon_1^2 + \epsilon_4^2$,
$\varphi_4 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2$

$$\Gamma_0 = \left\{ \underline{\epsilon} \text{ is given by (15) and (16)} \mid \varphi_4 \neq 0, \left(\frac{\varphi_2}{\varphi_4}\right)^q = \frac{\varphi_1}{\varphi_4} \right\}. \quad (17)$$

Theorem (Sufficiency condition) : $f_{\underline{\epsilon}}(X)$ is a permutation on $\mathbb{F}_{Q^2}$ with boomerang uniformity 4 if $\underline{\epsilon} \in \Gamma_0$

Conjecture (Necessity condition) : $f_{\underline{\epsilon}}(X)$ a permutation on $\mathbb{F}_{Q^2}$ with boomerang uniformity 4 only if $\underline{\epsilon} \in \Gamma_0$

(2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials

[Kim, SM, Choe, Lee, Lee, Jo, DCC 2022]

Problem : Identify $\Gamma = \{\underline{\epsilon} \in \mathbb{F}_Q^4 \mid f_{\underline{\epsilon}}(X) \text{ a permutation on } \mathbb{F}_{Q^2}\}$.

Question : $\Gamma = \Gamma_0$?

$\mu_{Q+1} = \{x \in \mathbb{F}_{Q^2} \mid x^{Q+1} = 1\}$

Proposition : $f_{\underline{\epsilon}}(X)$ is a permutation of $\mathbb{F}_{Q^2}$ if and only if $g_{\underline{\epsilon}}(X) = X^{q+1} h_{\underline{\epsilon}}(X)^{Q-1}$ is a permutation of $\mu_{Q+1} \setminus \{1\}$ where $h_{\underline{\epsilon}}(X) = \epsilon_1 X^{q+1} + \epsilon_2 X^q + \epsilon_3 X + \epsilon_4$.

Idea of the proof :

- $f_{\underline{\epsilon}}(X) = X^{q+1} h_{\underline{\epsilon}}(X^{Q-1})$
- Since $\gcd(q+1, Q-1) = 1$, $X^{q+1} h_{\underline{\epsilon}}(X^{Q-1})$ is a permutation of $\mathbb{F}_{Q^2}$ if and only if $g_{\underline{\epsilon}}(X) = X^{q+1} h_{\underline{\epsilon}}(X)^{Q-1}$ is a permutation of $\mu_{Q+1} = \{x \in \mathbb{F}_{Q^2} \mid x^{Q+1} = 1\}$ [Zieve, 2009].
- $g_{\underline{\epsilon}}(1) = 1$.

(2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials

[Kim et *al.*, DCC 2022] :

- $\mu_{Q+1} \setminus \{1\} = \{\frac{y+1+\gamma}{y+\gamma} \mid y \in \mathbb{F}_Q\}$, $\gamma \in \mathbb{F}_{Q^2} \setminus \mathbb{F}_Q$, $\gamma^2 = \gamma + 1$.
- Γ is the subset of $\mathbb{F}_Q^4$ of all $\underline{\epsilon}$ such that equation

$$g_{\underline{\epsilon}}\left(\frac{y+1+\gamma}{y+\gamma}\right) = \frac{a+1+\gamma}{a+\gamma} \tag{18}$$

has a unique solution y in $\mathbb{F}_Q$ for every $a \in \mathbb{F}_Q$

(2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials

Linking to "cubic" equations : Introducing polynomials $R_{\underline{\epsilon}}$, $S_{\underline{\epsilon}}$ and $T_{\underline{\epsilon}}$ in $\mathbb{F}_Q[X]$ whose coefficients depends on $\underline{\epsilon}$, we prove

- When $((R_{\underline{\epsilon}}(a))^q + S_{\underline{\epsilon}}(a) = 0$, Equation (18) has two solutions in $\mathbb{F}_Q$.
- When $((R_{\underline{\epsilon}}(a))^q + S_{\underline{\epsilon}}(a) \neq 0$, Equation (18) has one solution in $\mathbb{F}_Q$ if and only if

$$Y^{q+1} + Y + U_{\underline{\epsilon}}(a) = 0 \tag{19}$$

has one solution in $\mathbb{F}_Q$ where
$Y = ((R_{\underline{\epsilon}}(a))^q + S_{\underline{\epsilon}}(a))^{-1}(y^q + R_{\underline{\epsilon}}(a))$ and $U_{\underline{\epsilon}}(a) = \frac{(R_{\underline{\epsilon}}(a)S_{\underline{\epsilon}}(a)+T_{\underline{\epsilon}}(a))^q}{(R_{\underline{\epsilon}}(a)^q+S_{\underline{\epsilon}}(a))^{q+1}}$.

(2) A proof of the bijectivity property of an "exceptional" cryptographic family of quadrinomials

Proposition :
$\Gamma = \left\{ \epsilon \in \mathbb{F}_Q^4 \mid \varphi_4 \neq 0, \left(\frac{\varphi_2}{\varphi_4}\right)^q = \frac{\varphi_1}{\varphi_4}, \mathrm{Tr}_1^m\left(\frac{\varphi_3}{\varphi_4}\right) = 1 + k \right\} \supsetneq \Gamma_0$

Theorem [Kim et *al.* 2022] :
$f_{\underline{\epsilon}}(X)$ a permutation on $\mathbb{F}_{Q^2}$ if and only if $\underline{\epsilon} \in \Gamma$.

Corollary [Kim, SM, Kim, Jo (2022)] : $f_{\underline{\epsilon}}(X)$ a permutation on $\mathbb{F}_{Q^2}$ with Boomerang uniformity 4 if and only if $\underline{\epsilon} \in \Gamma_0$.

Proposition [Kim, SM, Kim, Jo (2022)] :
$\Gamma \setminus \Gamma_0 = \left\{ \underline{\epsilon} \in \Gamma \mid \mathrm{Tr}_1^m\left(\frac{\varphi_2 \varphi_2{}^Q}{\varphi_4^2}\right) = 1 \right\}$.

Conclusion

Still much to do concerning the resolution of equations over finite fields !

Some open problems (amount many others) :

- Solve the equations in the form $F(x) + F(x+1) = b$, where F is a power functions of finite fields in characteristic 2.
- Study the boomerang uniformity of known functions F whose $\delta(F) = 4$.