Boolean functions for symmetric cryptography: immersion and insights

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Outline

- Preliminaries on p-ary functions and some applications
- The (practical) case of Boolean functions in cryptography
- The main mathematical problems in symmetric cryptography

Let q be a power of a prime p and r be a positive integer. The trace function $Tr_{q^r/q} : \mathbb{F}_{q^r} \to \mathbb{F}_q$ is defined as :

$$Tr_{q^r/q}(x) := \sum_{i=0}^{r-1} x^{q^i} = x + x^q + x^{q^2} + \cdots + x^{q^{r-1}}.$$

The trace function from $\mathbb{F}_{q^r} = \mathbb{F}_{p^n}$ to its prime subfield $\mathbb{F}_p$ is called the *absolute trace* function.

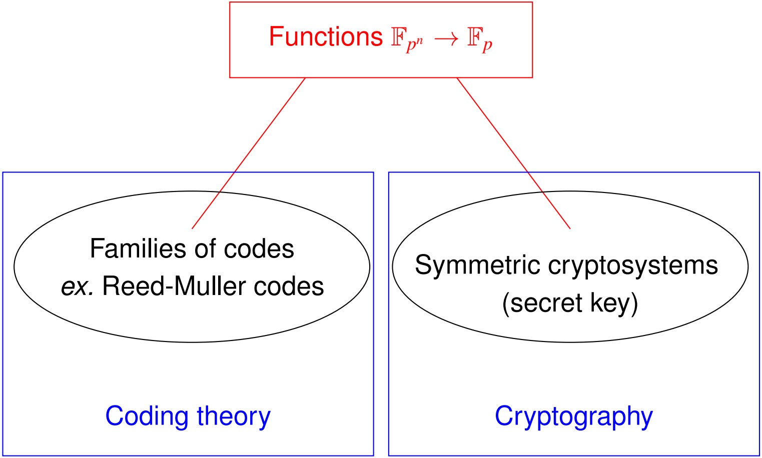
Background on functions over finite fields

Let $q = p^r$ where p is a prime.

- *A vectorial function $\mathbb{F}_q^n \to \mathbb{F}_q^m$ (or $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$) is called an (n, m)-q-ary function.*
- *When $q = 2$, an (n, m)-2-ary function will be simply denoted an (n, m)-function. They are called S-boxes (substitution-boxes). when they are used in a block cipher (in symmetric cryptography).*
- *A Boolean function is an $(n, 1)$-function, i.e. a function $\mathbb{F}_2^n \to \mathbb{F}_2$ (or $\mathbb{F}_{2^n} \to \mathbb{F}_2$).*

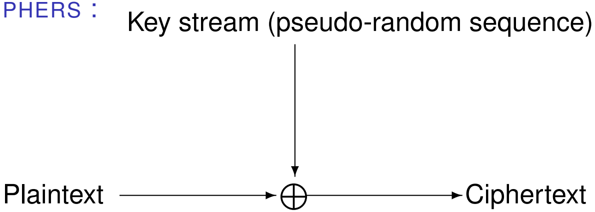
p-ary functions in cryptography and coding theory

- ☞ Functions from the finite field $\mathbb{F}_{p^n}$ to the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (p-ary functions) play an important role in coding theory and cryptography !



Cryptographic framework for Boolean functions

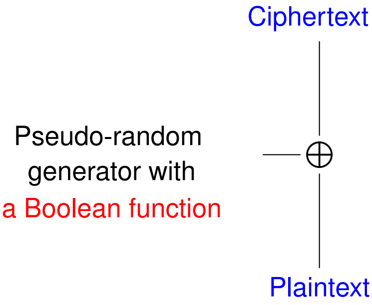
STREAM CIPHERS :



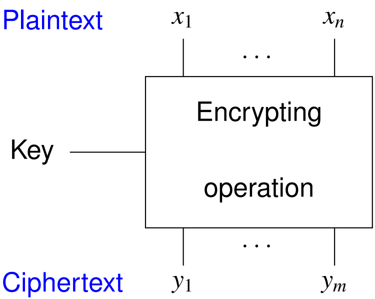
- To generate the key stream, we use models (eg. the combiner model, the filtered model) involving Boolean functions.
- The key stream has to follow properties related to the two fundamental principles introduced by Shannon : confusion and diffusion.
- The level of security of the cryptosystem against the known attacks can be quantified through some fundamental characteristics of the Boolean functions.

Boolean functions : cryptographic framework

Stream ciphers



Block ciphers (AES, DES, etc)



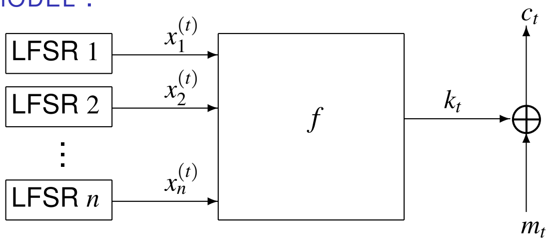
outputs of Boolean functions (depending on the key) over $x_1, \cdots, x_n$

Cryptographic framework for Boolean functions

The two models of pseudo-random generators with a Boolean function :

COMBINER MODEL :

m_t : plain text
c_t : cipher text
k_t : key stream

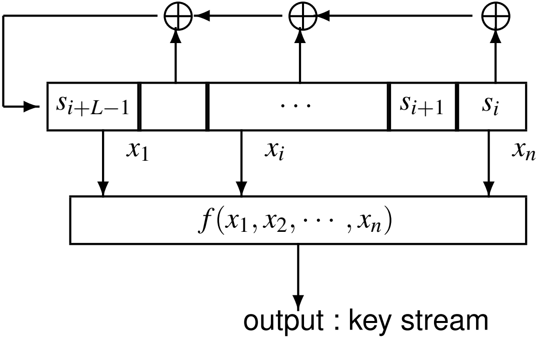


LFSR : Linear Feedback Shift Register

- A Boolean function combines the outputs of several LFSR to produce the keystream : **a combining (Boolean) function** f.
- The initial state of the LFSR's depends on a secret key.

Cryptographic framework for Boolean functions

FILTER MODEL :



- A Boolean function takes as inputs several bits of a single LFSR to produce the keystream : **a filtering (Boolean) function** f

  ☞ To make the cryptanalysis very difficult to implement, we have to pay attention when choosing the Boolean function : several recommendations (cryptographic criteria) !

Main cryptographic criteria for Boolean functions

• CRITERION 1 : To protect the system against distinguishing attacks, the cryptographic function f must be balanced, that is, its Hamming weight $wt(f) := \#\{x \in \mathbb{F}_{2^n}, f(x) \neq 0\}$ equals 2^{n-1}.

• CRITERION 2 : The cryptographic function must have a high algebraic degree to protect against the Berlekamp-Massey attack.

The Hamming distance $d_H(f,g) := \#\{x \in \mathbb{F}_{2^n} \mid f(x) \neq g(x)\}$.

• CRITERION 3 : To protect the system against linear attacks and correlation attacks, the Hamming distance from the cryptographic function to all affine functions must be large. i.e. high nonlinearity $\mathrm{nl}(f) := \min_{l \in A_n} d_H(f,l)$; l : affines functions.

• CRITERION 4 : To be resistant against correlation attacks on combining registers, a combining function f must be m-resilient where m is as large as possible. i.e. f must stay balanced if we fix at most m coordinates.

• CRITERION 5 : To be resistant against algebraic attacks, f must be of high algebraic immunity that is, close to the maximum $\lceil \frac{n}{2} \rceil$. Algebraic immunity of f : $\mathrm{AI}(f)$ is the lowest degree of any nonzero function g such that $f \cdot g = 0$ or $(1+f) \cdot g = 0$.
But this condition is insufficient because of Fast Algebraic Attacks !

Representations of p-ary functions

There is a unique representation of F from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^n}$: $F(x) = \sum_{i=0}^{2^n-1} a_i x^i$ with $a_i \in \mathbb{F}_{p^n}$. A unique univariate form of a p-ary function, called *trace representation*, is given by

$$f(x) = \sum_{j \in \Gamma_n} Tr_{p^{o(j)}/p}(A_j x^j) + A_{p^n-1} x^{p^n-1}$$

- Γ_n is the set of integers obtained by choosing the smallest element, called the *coset leader*, in each p- cyclotomic coset modulo $p^n - 1$;
- $o(j)$ is the size of the cyclotomic coset containing j (that is the smallest positive integer such that $jp^{o(j)} \equiv j \pmod{p^n - 1}$; $o(j)$ divides n ;
- $A_j \in \mathbb{F}_{p^{o(j)}}$ and $A_{p^n-1} \in \mathbb{F}_p$.

The *algebraic degree* of f : $deg(f) := max\{w_p(j) \mid A_j \neq 0\}$ where $w_p(j)$ is the number of nonzero entries in the p-ary expansion of j. For example, affine functions f : $d_{alg}(f) = 1$.

Background on Boolean functions : existence of the polynomial form

Any function $f : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ admits a unique representation :
$f(x) = \sum_{j=0}^{2^n-1} A_j x^j$; $A_j \in \mathbb{F}_{2^n}$;

- f is Boolean iff

$A_0, A_{2^n-1} \in \mathbb{F}_2$ and $A_{2j \ mod \ 2^n-1} = (A_{j \ mod \ 2^n-1})^2; 0 < j < 2^n - 1$

- $[1, 2^n - 2] = \cup_{r=1}^{c} \Gamma_r$;

$\Gamma_r = \{j_r \ mod \ 2^n - 1, 2j_r \ mod \ 2^n - 1, \cdots, 2^{o(j_r)-1} j_r \ mod \ 2^n - 1\}$

$$
\begin{aligned}
f(x) &= A_0 + A_{2^n-1} x^{2^n-1} + \sum_{r=1}^{c} \sum_{s=0}^{o(j_r)-1} A_{2^s j_r \ mod \ 2^n-1} x^{2^s j_r} \\
&= A_0 + A_{2^n-1} x^{2^n-1} + \sum_{r=1}^{c} \sum_{s=0}^{o(j_r)-1} (A_{j_r \ mod \ 2^n-1} x^{j_r})^{2^s} \\
&= A_0 + A_{2^n-1} x^{2^n-1} + \sum_{r=1}^{c} Tr_{2^{o(j_r)}/2}(A_{j_r \ mod \ 2^n-1} x^{j_r})
\end{aligned}
$$

where $A_0, A_{2^n-1} \in \mathbb{F}_2$, $A_{j_r \ mod \ 2^n-1} \in \mathbb{F}_{2^{o(j_r)}}$.

Background on Boolean functions : representation

Example : Let $n = 4$. $f : \mathbb{F}_{2^4} \to \mathbb{F}_2$,

$f(x) = \sum_{j \in \Gamma_4} Tr_{2^{o(j)}/2}(A_j x^j) + A_{2^4-1} x^{2^4-1}$;

Γ_4 is the set obtained by choosing one element in each cyclotomic class of 2 modulo $2^n - 1 = 2^4 - 1 = 15$. $C(j)$ the cyclotomic coset of 2 modulo 15 containing j.

$C(j) = \{j, j2, j2^2, j2^3, \cdots, j2^{o(j)-1}\}$ where $o(j)$ is the smallest positive integer such that $j2^{o(j)} \equiv j \pmod{2^n - 1}$.

The cyclotomic cosets modulo 15 are :

$C(0) = \{0\}$

$C(1) = \{1, 2, 4, 8\}$

$C(3) = \{3, 6, 12, 9\}$

$C(5) = \{5, 10\}$

$C(7) = \{7, 14, 11, 13\}$

We find $\Gamma_4 = \{0, 1, 3, 5, 7\}$

$f(x) =$

$Tr_{o(1)/2}(A_1 x^1) + Tr_{o(3)/2}(a_3 x^3) + Tr_{o(5)/2}(A_5 x^5) + Tr_{o(7)/2}(A_7 x^7) + A_0 + A_{15} x^{15}$

$= Tr_{4/2}(A_1 x) + Tr_{4/2}(A_3 x^3) + Tr_{2/2}(A_5 x^5) + Tr_{4/2}(A_7 x^7) + A_0 + A_{15} x^{15}$

where $A_1, A_3, A_7 \in \mathbb{F}_{2^4}$, $A_5 \in \mathbb{F}_{2^2}$ and $A_0, A_{15} \in \mathbb{F}_2$;

$Tr_{4/2} : \mathbb{F}_{2^4} \to \mathbb{F}_2$; $x \mapsto x + x^2 + x^{2^2} + x^{2^3}$;

$Tr_{2/2} : \mathbb{F}_{2^2} \to \mathbb{F}_2$; $x \mapsto x + x^2$.

Representations of p-ary functions

Viewed over $\mathbb{F}_p^n$, a p-ary function f has a representation as a unique multinomial in $x_1, \cdots, x_n$, where the variables x_i occur with exponent at most $p-1$. This is called the *multivariate representation* or algebraic normal form (ANF) of f :

$$f(x_1, \ldots, x_n) = \sum_{(j_1,\ldots,j_n)\in\mathbb{F}_p^n} a_{(j_1,\ldots,j_n)} \prod_{i=1}^{n} x_i^{j_i},$$

with coefficients $a_{(j_1,\ldots,j_n)} \in \mathbb{F}_p$. The degree of a monomial $\prod_{i=1}^n x_i^{j_i}$. is $j_1 + \cdots + j_n$.

The *algebraic degree* of f (denoted by $d_{alg}(f)$) is the global (total) degree of its multivariate representation, that is, the largest degree of all monomials in its ANF with a nonzero coefficient $a_{(j_1,\ldots,j_n)}$.

Example : A Boolean function $f(x) = f(x_1, \ldots, x_n)$ can be (uniquely) written as $f(x) = \bigoplus_{I\subseteq\{1,\ldots,n\}} a_I \left(\prod_{i\in I} x_i\right)$;

where " $\oplus$ " is the addition is made modulo 2 and a_I belongs to $\mathbb{F}_2$.

Background on Boolean functions

Example : Let $n = 3$.

x_1	x_2	x_3	$f(x_1,x_2,x_3)$	$g(x_1,x_2,x_3)$	$h(x_1,x_2,x_3)$
0	0	0	0	0	1
0	0	1	0	1	0
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	1	1	0
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	0	0

$f(x_1,x_2,x_3) = x_1 + x_2 + x_1x_2 + x_1x_3$; $d_{alg}(f) = 2$;
$g(x_1,x_2,x_3) = x_1 + x_2 + x_3 + x_1x_2 + x_2x_3 + x_1x_2x_3$; $d_{alg}(g) = 3$;
$h(x_1,x_2,x_3) = 1 + x_1 + x_2 + x_3$; $d_{alg}(h) = 1$.

Symmetric encryption

$F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^m}$

- **How does symmetric cryptography work ?**
  Attack on a cryptographic system $\Rightarrow$ cryptographic parameter of F $\Rightarrow$ cryptographic criterion

  Main Goal : Design "optimal" cryptographic functions to resist attacks !

- two big problems :

1. it is also necessary to study the properties of the functions which satisfy several of them cryptographic criteria (and not just one) compromise $\hookrightarrow$ to be found mathematically !
2. space too large (doubling exponential 2^{2^n} if $m = 1$) $\hookrightarrow$ need math !

☞ mathematical work to do for each cryptographic property :

a studies its algebraic properties of the function F satisfying cryptographic properties ;

b provides an efficient mathematical characterisation of each cryptographic criterion ;

c design functions F (given by their algebraic representations), which are optimal with respect to a cryptographic property.

And that is not enough : they must also be classified (think of the notions of equivalence) !

☞ **Key tool to study f :** need discrete Fourier theory

$f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ (resp. $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$), p prime

The discrete Hadamard Walsh (Fourier) transform W_f de f (resp. de F) :

Symmetric Cryptography

☞ **Key tool to study f : need discrete Fourier theory**

$f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ (resp. $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$), p prime

The discrete Hadamard Walsh (Fourier) transform W_f of f (resp. of F) :

$$W_f(a) := \sum_{x \in \mathbb{F}_{p^n}} \zeta_p^{f(x) - Tr_{p^n/p}(ax)}$$

$$W_F(a,b) := \sum_{x \in \mathbb{F}_{p^n}} \zeta_p^{Tr_{p^m/p}(bF(x)) - Tr_{p^n/p}(ax)}$$

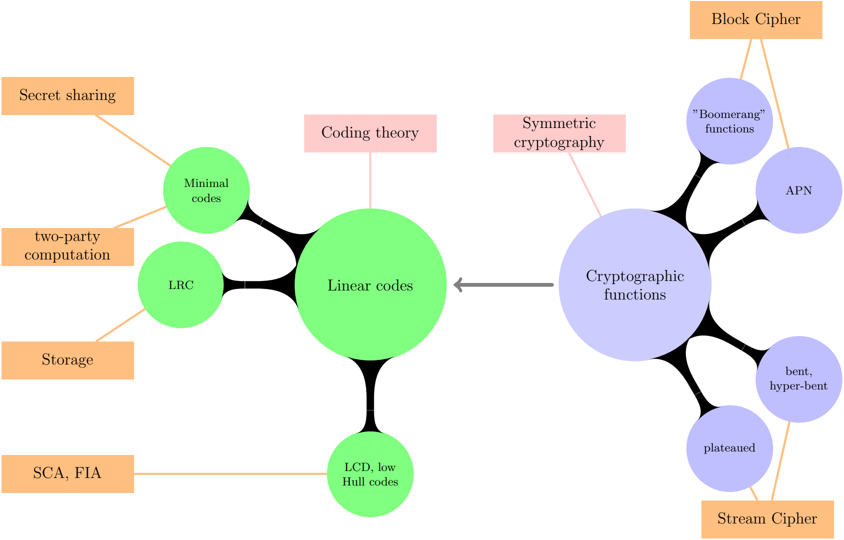
where $(a,b) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^m} \setminus \{0\}$

- The p-ary functions $F_b : x \mapsto b \cdot F(x) := Tr_{p^m/p}(bF(x))$, $b \in \mathbb{F}_{p^m}$ where $b \neq 0$ (F_0 is the null function) are called the *components* of F
- W_f (resp. W_F) is with values in the cyclotomic field $\mathbb{Q}(\zeta_p)$ where $\zeta_p = exp\left(\frac{2\pi i}{p}\right)$ is a p-th primitive root of the unit.

- There is an algorithm for calculating W_f (resp. W_F) but that is not enough ! (complexity too high $\hookrightarrow$ evaluate W_f mathematically !)
- Parseval identity : $\sum_{b\in\mathbb{F}_{p^n}} |W_f(b)|^2 = p^{2n}$
- Note that the notion of a Walsh transform refers to a scalar product, it is convenient to choose the isomorphism such that the canonical scalar product "$\cdot$" in $\mathbb{F}_{p^n}$ coincides with the canonical scalar product in $\mathbb{F}_{p^n}$, which is the trace of the product $b \cdot x := Tr_{p^n/p}(bx)$.
- Walsh transform of a very simple (but important function : Kloosterman sums on $\mathbb{F}_{2^m}$: $K_m(a) := \sum_{x\in\mathbb{F}_{2^m}} (-1)^{Tr_{2^m/2}(ax+\frac{1}{x})}$.

Extension of the theory of cryptographic Boolean functions to :

1. Vectorial Boolean functions
2. Functions in odd characteristic
3. Generalized functions



Block Cipher
Secret sharing
Coding theory
Symmetric cryptography
"Boomerang" functions
Minimal codes
APN
two-party computation
LRC
Linear codes
Cryptographic functions
Storage
bent, hyper-bent
plateaued
SCA, FIA
LCD, low Hull codes
Stream Cipher

Approaches and tools used to solve problems in this topic

Approaches : an algebraic approach, combinatoric approach, asymptotic approach, and geometric approach.

Mathematical tools :

- discrete Fourier/Walsh transforms
- polynomials over finite fields (polynomials, Linearized polynomials, permutation polynomials, involutions, Dickson polynomials, polynomials e- to-1, etc.)
- functions over finite fields (symmetric functions, quadratic forms, etc.)
- tools from algebraic geometry (algebraic, elliptic curves, hyper-elliptic curves, etc.)
- finite geometry (oval polynomials, hyperovals, etc.)
- linear algebra and group theory
- tools from combinatorics
- tools from arithmetic number theory

The nonlinearity of p-ary functions (where $p = 2$)

The *nonlinearity* of f defined over $\mathbb{F}_2^n$ is the minimum Hamming distance to the set A_n of all affine functions :

$$nl(f) = \min_{g \in A_n} d(f, g),$$

where $d(f, g)$ is the Hamming distance between f and g, that is $d(f, g) := \#\{x \in \mathbb{F}_2^n \mid f(x) \neq g(x)\}$. The relationship between nonlinearity and Walsh spectrum of f is

$$nl(f) = 2^{n-1} - \frac{1}{2} \max_{\omega \in \mathbb{F}_2^n} |W_f(\omega)|.$$

By Parseval's identity $\sum_{\omega \in \mathbb{F}_2^n} W_f(\omega)^2 = 2^{2n}$, it can be shown that $\max\{|W_f(\omega)| : \omega \in \mathbb{F}_2^n\} \geq 2^{\frac{n}{2}}$ which implies that $nl(f) \leq 2^{n-1} - 2^{\frac{n}{2}-1}$.

Let n be an even integer. An n-variable Boolean function is said to be bent if the upper bound $2^{n-1} - 2^{n/2-1}$ on its nonlinearity $nl(f)$ is achieved with equality.

- ☞ Bent Boolean functions function f defined over $\mathbb{F}_{2^n}$ exist only when n is even !
- ☞ The notion of bent function was introduced by [Rothaus 1976] and attracted a lot of research of more than four decades. Such functions are extremal combinatorial objects with several application areas, such as coding theory, maximum length sequences, and cryptography !
- ☞ f is bent over $\mathbb{F}_{2^n}$ if and only if, $W_f(\omega) \in \{2^{\frac{n}{2}}, -2^{\frac{n}{2}}\}$, for all $\omega \in \mathbb{F}_{2^n}$ ([Dillon 1974]).

☞ Linear Cryptanalysis [Matsui (1993)] $\Rightarrow$ nonlinearity $\mathrm{Nl}(F)$ de $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$,

$$\mathrm{Nl}(F) = \min_{b \in \mathbb{F}_{2^m}, b \neq 0} \{\mathrm{nl}(Tr_{2^m/2}(bF))\} = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n, b \in \mathbb{F}_2^m, b \neq 0} |W_F(a,b)|.$$

- ✭ The higher the value of $\mathrm{Nl}(F)$, the better resistance to linear cryptanalysis.
- ✭ When $n = m$ odd, $\mathrm{Nl}(F)$ is bounded by $2^{n-1} - 2^{\frac{n-1}{2}}$. The functions reaching this upper bound are the AB functions.
- ✭ When $m = 1$, $\mathrm{Nl}(F)$ is bounded by $2^{n-1} - 2^{\frac{n}{2}-1}$. The functions reaching this upper bound are the bent functions (n even).

$\hookrightarrow$ A very difficult parameter to study mathematically ! !