Classes of set-theoretical solutions to the pentagon equation

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An overview on the pentagon equation

The **pentagon equation (PE)** classically originates from the field of Mathematical Physics and belongs to an infinite family of equations, called the *polygon equations* **[Dimakis, Müller-Hossein - 2014]**.

Let V be a vector space over a field K.
A linear map $S: V \otimes V \to V \otimes V$ is a ***solution of the PE*** if it satisfies

$$S_{12} S_{13} S_{23} = S_{23} S_{12},$$

where $S_{12} = S \otimes \mathrm{id}_V$, $S_{23} = \mathrm{id}_V \otimes S$, $S_{13} = (\Sigma \otimes \mathrm{id}_V) S_{12} (\Sigma \otimes \mathrm{id}_V)$
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The PE has appeared in various contexts with different terminologies as:

- if H is a Hilbert space, a unitary operator on $H \otimes H$ is called multiplicative if it is a solution of the PE **[Baaj, Skandalis, 1993]**;
- for a fixed braided monoidal category $\mathcal{V}$, an arrow in $\mathcal{V}$ is named a fusion operator if it is a solution of the PE **[Street, 1998]**.

Set-theoretical solutions

Definition (Kashaev, Sergeev, 1998)

A *set-theoretical solution of the pentagon equation*, or *PE solution*, on a set X is a map $s : X \times X \to X \times X$ which satisfies the relation

$$s_{23}s_{13}s_{12} = s_{12}s_{23},$$

where $s_{12} = s \times \mathrm{id}_X$, $s_{23} = \mathrm{id}_X \times s$ and $s_{13} = (\mathrm{id}_X \times \tau)\, s_{12}\, (\mathrm{id}_X \times \tau)$ (here τ denotes the flip map $(x, y) \to (y, x)$).

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Let X be a finite set, s a PE solution on X, and S the linear operator from $K^{X \times X}$ into itself given by

$$S(f)(x, y) = f(s(x, y)),$$

for all $x, y \in X$. Then, S is a solution of the pentagon equation. Furthermore, if the map s is bijective, S is invertible.

Examples of solutions

(1) Let S be a semigroup and γ an idempotent endomorphism of S, i.e., $\gamma^2 = \gamma$. Then, the map $s(x, y) = (xy, \gamma(y))$ is a PE solution on S.

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(2) Let X be a set and f, g idempotent maps from X into itself such that $fg = gf$. Then, the map $s(x, y) = (f(x), g(y))$ is a PE solution on X that belongs to the class of P-QYBE solutions **[Catino, M., Stefanelli - 2020]**.

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(3) Let G be a group with *univocal factorization* (A, B), $p_1 : G \to A$, and $p_2 : G \to B$ the projection maps such that every $x \in G$ can be written as $x = p_1(x)\, p_2(x)$. Then, the map

$$s(x, y) = \left(p_2\left(yp_1(x)^{-1}\right)x,\ yp_1(x)^{-1}\right)$$

is a PE solution on G **[Zakrzewski - 1992]**.

Notation

Let X be a set, $\cdot : X \times X \to X$ a binary operation, and $\theta_x : X \to X$ a map, for every $x \in X$. The map $s(x, y) = (x \cdot y, \theta_x(y))$ is a **PE solution** on X if and only if

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1. $(X, \cdot)$ is a semigroup,
2. $\theta_x(y \cdot z) = \theta_x(y) \cdot \theta_{x \cdot y}(z)$,
3. $\theta_{\theta_x(y)}\theta_{x \cdot y} = \theta_y$,

for all $x, y, z \in X$.

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All the PE solutions on a group $(G, \cdot)$ can be determined by means of the normal subgroups of G **[Catino, M., Miccoli - 2020]**.

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A list of **202** PE solutions, up to isomorphism, on the semigroups of order 3 can be found in my Ph.D. thesis **[M. - 2021]**.

Involutive PE solutions

Theorem (Colazzo, Jespers, Kubat - 2020)

Let S be a semigroup and s an involutive PE solution on S, i.e., $s^2 = \mathrm{id}_{S\times S}$. Then,

$$S = E \times G \quad \& \quad s = s_E \times s_G,$$

where

- E is a left zero semigroup,
- G is an elementary abelian 2-group,
- $s_E(i,j) = (i, \theta_i(j))$ the unique bijective solution on E,
- $s_G(g,h) = (gh, h)$ the unique bijective solution on G.

Moreover, the converse is also true.

Idempotent PE solutions on monoids with central idempotents

Theorem (M. - 2023)

Let M be a monoid such that $\mathrm{E}(M) \subseteq \mathrm{Z}(M)$.
Let $\mu : M \to \mathrm{E}(M)$ an idempotent homomorphism such that, for every $x \in M$, $\mu(x) = e_x$, with e_x a right identity for x.

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Moreover, let $\{\theta_e : M \to M \mid e \in \operatorname{Im}\mu\}$ be a family of maps such that $\theta_1 = \mu$ and

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1. $\forall e \in \operatorname{Im}\mu, \forall x, y \in M, f = \mu(ex) \qquad \theta_e(xy) = \theta_e(x)\theta_f(y)$,
2. $\forall e, f \in \operatorname{Im}\mu \qquad \theta_e = \theta_e\theta_{ef}$,
3. $\forall e \in \operatorname{Im}\mu, \forall x \in M, f = \mu(x) \qquad \theta_{ef}\theta_e(x) = \theta_e(x)$.

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3. $\forall e \in \operatorname{Im}\mu, \forall x \in M, f = \mu(x) \quad \theta_{ef}\theta_e(x) = \theta_e(x)$.

Then, the map $s(x, y) = \left(xy, \theta_{\mu(x)}(y)\right)$ is an idempotent PE solution on M.

Conversely, every idempotent PE solution on M can be so constructed.

PE solutions on Clifford semigroups

A semigroup S is called *inverse* if, for each $a \in S$, there exists a unique $a^{-1} \in S$ satisfying

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$

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An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

Every Clifford semigroup S gives rise to the following PE solutions:

$$\mathcal{I}(a,b) = (ab, b), \qquad \mathcal{F}(a,b) = \left(ab, bb^{-1}\right), \qquad \mathcal{E}(a,b) = (ab, e),$$

where $e \in \mathrm{E}(S)$.

New classes of PE solutions

Definition (M., Stefanelli, Pérez-Calabuig - 2023)

Let s be a PE solution on a semigroup S. Then, s is said to be

- *idempotent-invariant* if $\theta_a(e) = \theta_a(f)$,
- *idempotent-fixed* if $\theta_a(e) = e$,

for all $a \in S$ and $e, f \in \mathrm{E}(S)$.

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Observe that the PE solution $\mathcal{E}(a, b) = (ab, e)$ is idempotent-invariant, while the PE solutions $\mathcal{I}(a, b) = (ab, b)$ and $\mathcal{F}(a, b) = \left(ab, bb^{-1}\right)$ are idempotent-fixed.

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We completely describe the idempotent-invariant PE solutions and provide a construction of idempotent-fixed PE solutions on Clifford semigroups.

E(S)-invariant PE solutions on a Clifford S

Theorem (M., Stefanelli, Pérez-Calabuig - 2023)

Let S be a Clifford semigroup. Consider

- ρ a congruence on S such that S/ρ is a group,
- $\mathcal{R}$ a system of representatives of S/ρ,
- $\mu : S \to \mathcal{R}$ a map such that

$$\forall a, b \in S \quad \mu(ab) = \mu(a)\,\mu(a)^{-1}\,\mu(ab) \quad \& \quad \mu(a) \in [a]_\rho.$$

Then, $s(a,b) = \left(ab, \mu(a)^{-1}\mu(ab)\right)$ is an E(S)-invariant PE solution on S.

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Then, $s(a, b) = \left(ab, \mu(a)^{-1}\,\mu(ab)\right)$ is an E(S)-invariant PE solution on S.

Conversely, if $s(a, b) = (ab, \theta_a(b))$ is an E(S)-invariant PE solution on S, then

- there exists a congruence $\rho_{(K,\tau)}$ such that $S/\rho_{(K,\tau)}$ is a group, where

$$K = \{a \in S \mid \forall\ e \in \mathrm{E}(S),\ e \leq a \quad \theta_e(a) \in \mathrm{E}(S)\} \quad \& \quad \tau = \mathrm{E}(S) \times \mathrm{E}(S),$$

- $\theta_e(S)$ is a system of representatives $S/\rho_{(K,\tau)}$,
- $\forall e \in \mathrm{E}(S), a, b \in S \quad \theta_e(ab) = \theta_e(a)\theta_e(a)^{-1}\theta_e(ab) \quad \& \quad (\theta_e(a), a) \in \rho_{(K,\tau)}$.

Moreover, $\theta_a(b) = \theta_e(a)^{-1}\theta_e(ab)$, for all $a, b \in S$.

PE solutions on groups

Corollary (Catino, M., Miccoli - 2020)

Let G be a group and $K \trianglelefteq G$. Moreover, consider

- R a system of representatives of G/K,
- $\mu : G \to R$ a map such that $\mu(a) \in Ka$, for every $a \in G$.

Then, the map $s(a, b) = (ab,\ \mu(a)^{-1}\mu(ab))$ for all $a, b \in G$, is a PE solution on G.

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Then, the map $s(a, b) = (ab,\ \mu(a)^{-1}\mu(ab))$ for all $a, b \in G$, is a PE solution on G.

Vice versa, if $s(a, b) = (ab, \theta_a(b))$ is a PE solution on G, then

- $K = \{a \in G \mid \theta_1(a) = 1\}$ is a normal subgroup of G,
- $\theta_1(G)$ is a system of representatives of G/K,
- $\theta_1(a) \in Ka$, for every $a \in G$,
- $\theta_a(b) = \theta_1(a)\theta_1(ab)^{-1}$, for all $a, b \in G$.

E(S)-fixed solutions on a Clifford S

Bearing that a Clifford semigroup S can be seen as a *strong semilattice of groups*, we give a method to construct an idempotent-fixed PE solution on S from solutions on each group.

Theorem (M., Stefanelli, Pérez-Calabuig - 2023)

Let $s^{[e]}(a,b) = \left(ab, \theta_a^{[e]}(b)\right)$ be a PE solution on G_e, for every $e \in \mathrm{E}(S)$. Moreover, for all $e, f \in \mathrm{E}(S)$, let $\epsilon_{e,f} : G_e \to G_f$ be maps such that $\epsilon_{e,f} = \varphi_{e,f}$ if $e \geq f$. If the following conditions are satisfied

$$\theta^{[h]}_{\epsilon_{ef,h}(ab)} = \theta^{[h]}_{\epsilon_{e,h}(a)\epsilon_{f,h}(b)},$$

$$\epsilon_{f,h}\theta^{[f]}_{\epsilon_{e,f}(a)}(b) = \theta^{[h]}_{\epsilon_{e,h}(a)}\epsilon_{f,h}(b),$$

for all $e, f, h \in \mathrm{E}(S)$ and $a \in G_e$ and $b \in G_f$, then the map

$$s(a,b) = \left(ab, \theta^{[f]}_{\epsilon_{e,f}(a)}(b)\right),$$

for all $a \in G_e$ and $b \in G_f$, is an idempotent-fixed PE solution on S.

Some questions

Recap:

- PE solutions on groups ✓
- Involutive PE solutions ✓
- Idempotent PE solutions on monoids with central idempotents ✓
- Idempotent-invariant PE solutions on Clifford semigroups ✓

Some problems:

1. Describe the idempotent-fixed PE solutions on Clifford semigroups.
2. Study other classes of PE solutions on Clifford semigroups.
3. Study idempotent, idempotent-fixed or idempotent-invariant PE solutions on specific semigroups.

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Thank you!