Algebraic structure of quadratic algebras and set-theoretic solutions of the Yang–Baxter equation

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1. INTRODUCTION
1.1 Structure algebra and monoid

Let X be a set and let $r : X \times X \to X \times X$ be a map.
Its structure algebra is

$$\begin{aligned} \mathcal{A}(X,r) &= K\langle x \in X\rangle / (\ xy = uv \text{ if } r(x,y) = (u,v),\ x,y \in X\ ) \\ &= K[M(X,r)] \text{ (monoid algebra)} \\ &= K\langle X \mid xy = uv \text{ if } r(x,y) = (u,v),\ x,y \in X\rangle \end{aligned}$$

The monoid

$$M = M(X,r) = \langle X \mid xy = uv \text{ if } r(x,y) = (u,v),\ x,y \in X\rangle$$

is the structure monoid of (X,r).

This algebra (monoid) is the associative ring (monoid) theoretic tool to investigate the map r and has attracted a lot of attention in case r satisfies the braided relation, i.e. (X,r) is a set-theoretic solution of the Yang-Baxter equation.

General Aim:
the link between r and the algebraic structure of $\mathcal{A}(X,r)$.

1.2 Set theoretic solutions of the Yang–Baxter equation (YBE)

(X, r) is a set-theoretic solution of the YBE if

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where $r_{12} = r \times \mathrm{id}$ and $r_{23} = \mathrm{id} \times r$ are maps $X^3 \to X^3$.
Write

$$r(x, y) = (\lambda_x(y), \rho_y(x)).$$

One says that (X, r) is:

- finite if $|X| < \infty$.
- bijective if r is a bijective map.
- involutive if $r^2 = \mathrm{id}$.
- idempotent if $r^2 = r$.
- left (respectively right) non-degenerate if all the maps λ_x (respectively ρ_y) are bijective.
- non-degenerate if (X, r) is left and right non-degenerate.

1.3 General outline of lectures

In this series of three lectures we explain some intriguing relationship between the algebraic structure of the structure algebras $\mathcal{A}_K(X, r)$ and the finite (mainly left non-degenerate) set-theoretic solutions (X, r) of the Yang-Baxter equation. We focus on the following ring theoretical properties of such algebras:

- left/right Noetherian
- prime
- semiprime
- representability
- Gelfand-Kirillov dimension
- the dimension of the degree 2-part of the algebra
- construction of such algebras
- What if r is not a set-theoretic solution?

Also attention will be given to the methods used: a combination of ring-, semigroup- and group techniques.

If (X, r) is a set-theoretic solution of the Yang-Baxter equation then $M(X, r)$ is called a Yang-Baxter monoid and $K[M(X, r)]$ is called a Yang-Baxter algebra.

1.4 Motivation

Monoid algebras and quadratic algebras arise in a variety of areas, including

- (non-)commutative ring theory
- (non-)commutative geometry
- algebras of low dimension
- quantum groups and Hopf algebras
- close relationship with group algebras
- via the solutions of the YBE widening into other fields, including mathematical physics

1.5 Some known general results/problems, what to expect?

Theorem (Zelmanov (1977))

K a field, S a cancellative semigroup. The following conditions are equivalent:

1. *$K[S]$ satisfies a polynomial identity, i.e. there exists $0 \neq f(X_1, \ldots, X_n) \in K\langle X_1, \ldots, X_n\rangle$ so that $f(r_1, \ldots, , r_n) = 0$ for all $r_1, \ldots, r_n \in K[S]$.*
2. *S has a group of quotients G so that $K[G]$ satisfies a PI.*
3. *(Passman) G is abelian-by-finite and char$(K) = 0$, or G is (finite p-group)-by-(abelian-by-finite) group and char$(K) = p > 0$.*

If a semigroup ring $K[S]$ of a semigroup S over a ring K is (right) Artinian, then the same is true for K and S is finite. These conditions on K and S are also sufficient for $K[S]$ to be (right) Artinian for a semigroup S with identity (for group algebras I. Connell).

It is known that in this case the following conditions are equivalent:

- $K[S]$ is semiprime
- $K[G]$ is semiprime
- $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) = p > 0$ and the torsion part of the finite conjugacy centre $\Delta(G)$ of G is a p'-group.

Open Problem

When is a monoid algebra $K[S]$ Noetherian?

Even for group algebras $K[G]$ the answer is unknown.

Some known results for group algebras $K[G]$.

1. $K[G]$ is left noetherian if and only if $K[G]$ is right noetherian (because of the involution $g \mapsto g^{-1}$).
2. If G is polycyclic-by-finite then $K[G]$ is Noetherian (easy, comes down to $R[x, x^{-1}, \sigma]$, skew Laurent polynomial algebra, is Noetherian if R is Noetherian).
3. If G is torsion-free and polycyclic-by-finite then $K[G]$ is a domain (not easy and proved via some homological properties, Farkas-Snider 1976, Cliff 1980).
4. If G is polycyclic-by-finite then $K[G]$ is Noetherian prime maximal order if and only if $\Delta^+(G) = \{1\}$ and G is dihedral free (for example if G is torsion-free). (K. Brown 1985, 1988, E. Jespers-P. Smith 1985)
5. If G is finitely generated torsion-free abelian-by-finite then $K[G]$ is a Noetherian PI maximal order and all its height one prime ideals P are generated by a normal element, i.e. $P = K[G]\alpha = \alpha K[G]$. So it behaves as a (non-commutative) UFD. (K. Brow 1985, 1988)

Known result for monoid algebras $K[S]$

Theorem (Jespers-Okniński)

Let S be a submonoid of a polycyclic-by-finite group. The following properties are equivalent:

1. *S satisfies the ascending chain on right ideals,*
2. *$K[S]$ is right Noetherian,*
3. *$S \subseteq G = SS^{-1}$, H a subgroup of finite index in G, $[H, H] \subseteq S$ and $S \cap H$ is finitely generated,*
4. *$K[S]$ is left Noetherian,*
5. *S satisfies the ascending chain on left ideals.*

So, for such monoids, the answer is left-right symmetric.

Interested reader: Jespers, Okniński, Noetherian Semigroup Algebras, Springer, Series: Algebra and Applications, 2007.

When is an algebra embedded into a matrix algebra?

Theorem (Ananin)

Any finitely generated right Noetherian PI algebra over a field is representable, i.e. embedded into a matrix algebra over a field.

2. Construction of quadratic algebras with "nice" properties via set-theoretic solutions of YBE
2.1 I-type monoids

An exciting link between quadratic algebras and set-theoretic solutions of the Yang-Baxter equation has been initiated by Gateva-Ivanova and Van den Bergh (1998) (and also Etingof, Schedler, and Soloviev (1999)).

A monoid S generated by a set $X = \{x_1, \dots, x_n\}$ is said to be of (left) I-type if there exists a bijection

$$v : \mathrm{FaM}_n \to S$$

such that, for all $a \in \mathrm{FaM}_n = \langle u_1, \dots, u_n \rangle$, the free abelian monoid of rank n,

$$v(1) = 1 \quad \text{and} \quad \{v(u_1 a), \dots, v(u_n a)\} = \{x_1 v(a), \dots, x_n v(a)\}.$$

Theorem (Gateva-Ivanova, Van den Bergh 1998)

A monoid M generated by $X = \{x_1, \ldots, x_n\}$ is of left I-type if and only if there exists a mapping $r : X^2 \to X^2$ so that

1. $r^2 = id$,
2. *r non-degenerate,*
3. *(X, r) is a set-theoretic solution of the YBE.*
4. *(Jespers-Okniński)* $M = \{(a, \lambda_a) \mid a \in \mathrm{FaM}_n\} \subseteq \mathrm{FaM}_n \rtimes \mathrm{Sym}_n$

Furthermore, M has a group of fractions

$$\begin{aligned} G = G(X, r) &= \mathrm{gr}(X \mid xy = \lambda_x(y)\rho_y(x),\ x, y \in X) \\ &\subseteq \{(a, \lambda_a) \mid a \in \mathrm{Fa}_n\} \subseteq \mathrm{Fa}_n \rtimes \mathrm{Sym}_n \end{aligned}$$

which is torsion-free abelian-by-finite,
where Fa_n is the free abelian group of rank n.

As a consequence one can show that often M can be decomposed as products of monoids and groups of the same type but on less generators (many such groups are poly-infinite cyclic). Minimal prime ideals of M, and height one primes of $K[M]$ are principal and generated by a normal element; all semiprime ideals of M are described. The latter yields an ideal chain of S with factors that are semigroups of matrix type over cancellative semigroups.

Theorem (Gateva-Ivanova and Van den Bergh 1998)

$K[M(X,r)] = K\langle X\rangle/(xy - \lambda_x(y)\rho_y(x) \mid x, y \in X)$

- *is a Noetherian PI domain,*
- *has GK-dimension $|X|$,*
- *is a maximal order*
- *all height one primes that intersect M non-trivially are generated by a normal element.*

Hence, this algebra "shares" many properties with commutative polynomial algebras.

Some background on Gelfand-Kirillov dimension

Let R be a finitely generated K-algebra over a field K. The Gelfand-Kirillov dimension $GKdim(R)$ measures the rate of growth of R (in terms of generating sets). For monoid algebras $K[M]$ it measures the rate of growth of M.

Let V be a finite dimensional subspace of R that generates R as an algebra. Put $R_n = \sum_{i=0}^{n} V^i$, a finite dimensional subspace.

$$GKdim(R) = \lim\sup_{n\to\infty} \left(\frac{\log \dim_K R_n}{\log n} \right).$$

It is independent of the choice of V.

- $GKdim(R) = 0$: R is finite dimensional.
- $GKdim(R) < \infty$: $dim_K(R_n) \leq n^m$ for sufficiently large n
- $GKdim(R) \in \{0, 1, 2, r \mid r \in \mathbb{R}\} \cup \{\infty\}$
- $GKdim(R[x_1, \ldots, x_n]) = GKdim(R) + n$

- if R satisfies a PI then $GKdim(R) < \infty$
- if R satisfies a PI and R is right Noetherian then $GKdim(R) = clKdim(R)$
- if S is a finitely generated cancellative semigroup then the following conditions are equivalent:
  - $GKdim(K[S]) < \infty$
  - S has a group of quotients G and $GKdim(K[G]) < \infty$
  - S has a group of quotients that is nilpotent-by-finite. (Gromov 1981 and Grigorchuk 1988).

  Moreover, in these cases $GKdim(K[S]) = GKdim(K[G])$.

2.2 Drinfel'd problem

V. G. Drinfel'd. (On some unsolved problems in quantum group theory. In Quantum groups (Leningrad, 1990), volume 1510 of Lecture Notes in Math., pages 1–8. Springer, Berlin, 1992):

"Maybe it would be interesting to study set-theoretical solutions ....The only thing I know about set-theoretical solutions is the following couple of examples."

Example 1 (V.V.Lyubashenko). If $r(x, y) = (f(y), g(x))$ then r is a set-theoretic solution of the YBE if and only if $fg = gf$.

Example 2: If $r(x, y) = (x, x \circ y)$ for some operation $\circ$ on X then being a solution of the YBE is equivalent to the following distributivity identity: $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$."

Drinfel'd Problem

Determine and classify all (finite) set-theoretic solutions of the Yang-Baxter equation on a set X:

2.3 Examples of set-theoretic solutions of YBE

X a set.

(1) If $r(x,y)=(y,x)$ for $x,y\in X$ (all $\lambda_x=\rho_x=\mathrm{id}$) then $K[M(X,r)]=K[X]$ commutative polynomial algebra.
It is a unique factorisation domain and it only is Noetherian if X is finite.

(2) If $r(x,y)=(x,y)$ (all λ_x and ρ_x are constant maps onto $\{x\}$) then $K[M(x,r)]=K\langle X\rangle$ the free algebra on X.
Only left or right Noetherian if $|X|=1$. This is a degenerate example.

(3) If $X=\{1,2,3\}$ and $r(x,y)=(y,\rho_y(x))$ for $x,y\in X$, where $\rho_1=(2\ 3)$, $\rho_2=(1\ 3)$ and $\rho_3=(1\ 2)$ then (X,r) is a non-degenerate bijective solution (with $r^3=\mathrm{id}$) and

$$K[M(X,r)]\cong K\langle x,y,z\mid xy=yz=zx \text{ and } zy=yx=xz\rangle$$

is Noetherian and PI, but not semiprime (let alone domain).

(4) If $|X| > 1$ then $r(x, y) = (y, y)$ is an idempotent solution, $K[M(X, r)]$ is left Noetherian and it only is right Noetherian if $|X| = 1$.
Note that $(x^{2d} - y^{2d})M = x^{2d} - y^{2d}$ and thus $\sum_{d \geq 1}(x^{2d} - y^{2d})$ is a nilpotent right ideal that is not finitely generated. So this algebra is not semiprime, not right Noetherian. But it is left Noetherian.

(5) $K[M(X, r)]/(xy \mid x, y \in X) \cong K + \oplus_{x \in X} K\overline{x}$ is a commutative algebra with large radical $J = \oplus_{x \in X} K\overline{x}$ and this is only finitely generated when X is finite .

To obtain "nice" properties on Yang-Baxter algebras (for example being left/right Noetherian) one needs to impose some restrictions on the solutions, restrictions on the λ's and/or ρ's.

2.4 Extending a solution to the Yang-Baxter monoid

Theorem (Gateva-Ivanova, Majid 2008)

- *the maps λ_x and ρ_y can be extended to the monoid $M = M(X, r)$ so that $a \circ b = \lambda_a(b) \circ \rho_b(a)$ for all $a, b \in M$.*
- *$r_M : M^2 \to M^2 : (a, b) \mapsto (\lambda_a(b), \rho_b(a))$ also is a solution of the YBE, called the associated solution. Obviously, r_M restricts to r.*
- *r_M is bijective non-degenerate if and only if r is bijective non-degenerate.*

2.5 Bijective solutions

Theorem (Castelli, Catino and Stefanelli 2021 and Colazzo, Jespers, Van Antwerpen, Verwimp 2022)

*Let (X, r) be a finite left non-degenerate solution of the YBE.
Then, r is bijective if and only if r is right non-degenerate.*

For a proof of the necessity Castelli, Catino and Stefanelli used q-cycle sets and and Cedo, Jespers, Van Antwerpen, Verwimp used semitrusses to prove the equivalence.

2.6 From now we only consider left non-degenerate solutions

Assumption:
$r : X^2 \to X^2 : (x, y) \mapsto (\lambda_x(y),\ \rho_y(x))$ is a finite left non-degenerate solution of the YBE
i.e. X is finite and all λ_x bijective,
and $M = (M(X, r), \circ)$ is its Yang-Baxter monoid.

Derived solution

$r : X^2 \to X^2 : (x, y) \mapsto (\lambda_x(y), \rho_y(x))$ left non-degenerate solution YBE

Its derived solution of the YBE

$$s : X^2 \to X^2 : \qquad (x, \lambda_x(y)) \mapsto (\lambda_x(y), \lambda_{\lambda_x(y)}(\rho_y(x)))$$
$$s : X^2 \to X^2 : \qquad (x, y) \mapsto (y, \lambda_y \rho_{\lambda_x^{-1}(y)}(x)) = (x, \sigma_y(x)).$$

with associated Yang-Baxter monoid

$$A = A(X, r) = M(X, s) = \langle a_1, \ldots, a_n \mid a_i + a_j = a_j + \sigma_{a_j}(a_i)) \rangle.$$

Note s encodes "behaviour" of r^2:

$$(x, y) \overset{r}{\mapsto} (\lambda_x(y), \rho_y(x)) \overset{r}{\mapsto} (\lambda_{\lambda_x(y)}(\rho_y(x), \rho_{\rho_y(x)}(\lambda_x(y))).$$

Focus only on first coordinates

$$(x, \lambda_x(y)) \overset{s}{\mapsto} (\lambda_x(y), \lambda_{\lambda_x(y)}(\rho_y(x))).$$

Lemma

For all $a \in A$:

$$A + a \subseteq a + A.$$

1. *Right ideals of A are two-sided ideals of A.*
2. *If r is also right non-degenerate (so bijective) then $A + a = a + A$ for all $a \in A$.*
   So elements of A are normal elements.
3. *If r is involutive (i.e. $r^2 = \mathrm{id}$) then A is a free abelian monoid with basis X.*

Theorem (Cedo-Jespers-Verwimp 2021)

There exists a bijective map $\pi : M(X,r) \to A(X,r)$, with $\pi(x) = x$ for $x \in X$, so that (we identify A with M)

1. $\lambda : (M, \circ) \to Aut(A, +) : a \mapsto \lambda_a$, *monoid homomorphism,*
2. $\rho : (M, \circ) \to Map(A, A) : a \mapsto \rho_a$ *monoid anti-homomorphism,*
3. $M(X,r) \to A(X,r) \rtimes Im(\lambda) : a \mapsto (a, \lambda_a)$ *monoid embedding.*

In particular, the size of M and A are the same and thus

$$GKdimK[M] = GKdimK[A] \leq |X| < \infty.$$

Warning: one can not extend this theorem to groups $G(X,r)$ except if r is bijective (Lu, Yan, Zhu).

3.2 Second main step to deal with Noetherian problem: the derived solution and $K[A]$

X a finite set , r a solution with each $\lambda_x = \mathrm{id}$. Thus $r = s$ and

$$A = A(X,r) = \langle x \in X \mid x + y = y + \sigma_y(x) \rangle$$

For all $a, b \in A$,

$$a + b = b + \sigma_b(a)$$

with monoid antihomomorphism

$$\sigma : (A,+) \to End(A,+) \text{ and } \sigma_{a+b} = \sigma_b \circ \sigma_a$$

and the finite monoid

$$\mathcal{C} = \mathcal{C}(A) = \{\sigma_a \mid a \in A\} = \langle \sigma_x \mid x \in X \rangle$$

and, for some large enough v

$$\sigma_a \circ \mathcal{C} \subseteq \mathcal{C} \circ \sigma_a \text{ and } \sigma_x^v = \sigma_{vx} = \sigma_{vx}^2$$

So right ideals are two-sided ideals and σ_x^v is idempotent.

- $B = B(v) = \{m_1 v x_1 + \cdots + m_n v x_n \mid m_1, \ldots, m_n \geq 0\}$ is a monoid.
- $A = B(v) + F(v)$ with $F(v) = \{m_1 x_1 + \cdots + m_n x_n \mid v > m_1, \ldots, m_n \geq 0\}$, A is a finite left module over $B(v)$.
- Put $\sigma_j = \sigma_{y_j}$, $X(v) = \{y_1 = vx_1, \ldots, y_n = vx_n\}$.
- $B(v) = A(X(v), s_{X_v})$ with s_{X_v} induced solution from s.
- $\sigma_{j_l} \cdots \sigma_{j_k} \sigma_{j_l} = \sigma_{j_l} \cdots \sigma_{j_k}$
- $\mathcal{C}(B(v)) = \{\sigma_{j_1} \cdots \sigma_{j_k} \mid 1 \leq j_1, \ldots, j_k \leq n\} \cup \{\mathrm{id}\}$ is a band, semigroup of idempotents.
- Put $t_k = y_1 + \cdots + y_k$, $B_{\kappa(t_k)} = \langle y_{\kappa(i)} \mid 1 \leq i \leq k \rangle$ and $T = \{\kappa(t_k) \mid 1 \leq k \leq n,\ \kappa \in \mathrm{Sym}_n\}$.
- Each $w \in B$ can be written $w = w_t + t$ with $t \in T$ and $w_t \in B_t$. Also $\sigma_w = \sigma_t$.
- For $a, b \in B_t$: $a + b + t = b + a + t$. Hence each $B_t + t$ is an abelian semigroup.
- $B(v) = \{0\} \cup \bigcup_{t \in T} (B(v)_t + t)$, a finite union of abelian semigroups.
- $\kappa(t_k) + y_{\kappa(l)} = y_{\eta(l')} + \eta(t_k)$.

- $K[B_t + t]$ is left $K[B_t]$ module (with $a \cdot (d+t) = a + d + t$). It is a cyclic module over $K[B_t]/[K[B_t],, K[B_t]]$ a commutative finitely generated ring and hence a Noetherian module.
- Definine $R_k = \sum_{\kappa \in \mathrm{Sym}_n} K[B_{\kappa(t_k)} + \kappa(t_k)]$

Theorem (Colazzo, Jespers, Van Antwerpen, Verwimp 2022)

- *$B_n \subseteq B_n \cup B_{n-1} \subseteq \cdots \subseteq B_n \cup \cdots \cup B_2 \cup B_1 \subseteq B$, an ideal chain of $B(v)$.*
- *A satisfies the ascending chain condition on left ideals*
- *$\{0\} \subseteq K[B_n] \subseteq K[B_n] + K[B_{n-1}] \subseteq \cdots \subseteq K[B_n] + \cdots + K[B_2] + K[B_1] \subseteq K[B]$, an ideal chain with each factor a Noetherian left $K[B]$ module that is a finite sum of commutative rings (in particular a PI-ring).*

Corollary (Colazzo, Jespers, Van Antwerpen, Verwimp)

If $|X| = n$ then $K[A]$ is a left Noetherian PI-ring with $\mathrm{GKdim} K[A] \leq n$.

Corollary (Colazzo, Jespers, Van Antwerpen, Verwimp)

Let (X, r) be a finite left non-degenerate solution. Then, $K[A \rtimes Im(\lambda)]$ is a finitely generated left Noetherian PI-algebra and thus a representable algebra. In particular, $K[M(X, r)]$ is a representable algebra and $M(X, r)$ is a linear semigroup, i.e. a subsemigroup of the multiplicative semigroup $M_n(F)$ with F a field.

Corollary

Let (X, r) be a finite left non-degenerate solution. The mapping $R \mapsto R^e = \{(a, \lambda_a) \mid a \in R\}$ is a bijection between the right ideals of $A(X, r)$ and those of $M(X, r)$. Thus, $M(X, r)$ satisfies the ascending chain condition on right ideals.

3.3 Third main step to deal with Noetherian problem

Theorem (Okniński)

Assume S is a finitely generated monoid with an ideal chain

$$\emptyset = S_{m+1} \subseteq S_m \subseteq S_{m-1} \subseteq \cdots \subseteq S_1 \subseteq S_0 = S$$

such that each factor S_j/S_{j+1} (for $0 \leq j \leq m$) is either power nilpotent or a uniform subsemigroup of a Brandt semigroup. If K is a field and S satisfies the ascending chain condition on right ideals and GKdim$K[S]$ is finite then $K[S]$ is right Noetherian.

A semigroup T with zero θ is power nilpotent if $T^m = \{\theta\}$ for some $m > 0$.
A completely 0-simple semigroup $\mathcal{M}^0(G, n, m, P)$ over a group G is a semigroup of $n \times m$- matrices with at most one nonzero entry in $g \in G$ at spot (i, j) , denoted (g, i, j), and with multiplication

$$(g, i, j)\,(h, k, l) = (gp_{jk}h, i, l)$$

where P is an $m \times n$-matrix with entries in $G^\theta = G \cup \{\theta\}$.

If $n = m$ and P is the identity matrix I this is a Brandt semigroup. A subsemigroup T of $\mathcal{M}^0(G, k, k, I)$ is said to be uniform if each $\mathcal{H}$-class (i.e., all the matrices with non-zero entries in a fixed (i, j)-spot) of $\mathcal{M}^0(G, k, k, I)$ intersects non-trivially T and the maximal subgroups of $\mathcal{M}^0(G, k, k, I)$ are generated by their intersection with T .

Let K be a field. Let $(S, \cdot)$ be a semigroup and I an ideal of S then $K[I]$ is an ideal of $K[S]$ and

$$K[S]/K[I] = \mathrm{vect}_K(S \setminus I).$$

In this algebra all elements of I "become" 0.
Let $S/I = (S \setminus I) \cup \{\theta\}$ with multiplication $\circ$ defined by

$$s \circ t = s \cdot t \quad \text{if } s, t \in (S \setminus I) \text{ and } s \cdot t \in (S \setminus I),$$

$$\text{otherwise} \quad s \circ t = \theta.$$

This is a semigroup, called the Rees factor of S by I, so that

$$K[S]/K[I] = K[S/I]/K\theta := K_0[S/I],$$

called the contracted semigroup algebra of S/I.

Background on linear semigroups

Let $A = M_n(F)$ be a matrix algebra over a field F and $(A, \cdot)$ its multiplicative semigroup. Put $A_i = \{a \in A \mid rk(a) \leq i\}$, a subsemigroup, for $0 \leq i \leq n$. Then

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

are the only ideals of the monoid A and

$$A_i / A_{i-1} \cong \mathcal{M}^0(GL_i(F), X_i, Y_i; Q_j),$$

where Y_i are all matrices if rank i which are in the reduced row elementary form and $X_i = Y_i^t$ and $Q_i = (q_{yx})$ is defined for $x \in X_i$, $y \in Y_i$ by $q_{yx} = yx$ if yx is of rank i and θ otherwise.
The groups $GL_i(F)$ are the maximal subgroups of $(A, \cdot)$.

Theorem (Okniński)

Let $S \subseteq A = M_n(F)$ be a linear semigroup. Write $S_i = S \cap A_i$, the matrices of S of rank at most i. Let T_i be the matrices in S_i so that $S^1 a S^1$ does not intersect maximal subgroups of A contained in $A_i \setminus A_{i-1}$. Then we have an ideal chain of S:

$$S_0 \subseteq T_1 \subseteq S_1 \subseteq T_1 \subseteq S_1 \subseteq T_2 \subseteq \cdots \subseteq S_{n-1} = T_n \subseteq S_n = A$$

with the following properties:

1. $N_i = T_i / S_{i-1}$ *is a nilpotent ideal of* S/S_{i-1}.
2. $(S_i \setminus T_i) \cup \{\theta\} \subseteq M_i/M_{i-1}$ *is a 0-disjoint union of uniform subsemigroups* $U_\alpha^{(i)}$, $\alpha \in \mathcal{A}_i$, *of* M_i/M_{i-1} *that intersect different* $\mathcal{R}$*- and different* $\mathcal{L}$*-classes of* M_i/M_{i-1}*; moreover* N_i *does not interesect* $\mathcal{H}$*-classes of* M_i/M_{i-1} *intersected by* $S_i \setminus T_i$.
3. $U_\alpha^{(i)} U_\beta^{(i)} \subseteq N_i$ *for* $\alpha \neq \beta$*; moreover* $U_\alpha^{(i)} N_i,\ N_i U_\alpha^{(i)} \subseteq N_i$ *and* $U_\alpha^{(i)} N_i U_\alpha^{(i)} = \{\theta\}$ *in* M_i/M_{i-1}.

3.4 How to construct such a chain in $M(X,r)$?

Let $1 \leq i \leq n = |X|$. Let

$$M_i = \bigcup_{Y \subseteq X,\ |Y|=i} \{a \in M \mid a = y\, b(y), \text{ for each } y \in Y \text{ and some } b(y) \in M\}$$
$$= \text{all elements of } M \text{ left divisible by at least } i \text{ elements of } X$$

This yields an ideal chain in $M = M(X,r)$:

$$\emptyset = M_{n+1} \subseteq M_n \subseteq \cdots \subseteq M_1 \subseteq M_0 = M.$$

Refine the chain

$$M_{i+1} \subseteq B_i \subseteq U_i \subseteq M_i$$

such that

- B_i/M_{i+1} and M_i/U_i are power nilpotent (if M_i/M_{i+1} is power nilpotent then $B_i = U_i = M_i$).
- if M_i/M_{i+1} is not power nilpotent then U_i/B_i is a disjoint union of semigroups $S_1, \ldots, S_m$ such that $S_k S_l \subseteq M_{i+1}$ for $k \neq l$.
- each $(S_i \cup M_{i+1})/M_{i+1}$ is a uniform subsemigroup of a Brandt semigroup.

The intermediate ideals B_i and U_i are build with the use of the sets $Y, Z \subseteq X$, with $|Y| = |X| = i$,

$$M_{YZ} = \{(a, \lambda_a) \mid a \text{ left divisible precisely by elements of } Y \text{ and } \lambda_a(Z) = Y\}$$

If $Y \subseteq X$ and M_{YY} is a non-empty semigroup then there exists

$$m_Y = (a_Y, \mathrm{id}) \in M_{YY}.$$

Theorem (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

Let (X, r) be a finite left non-degenerate solution of the YBE and let K be a field.

1. *If for each $Y \subseteq X$ for which M_{YY} is a (non-empty) semigroup there exists a $d \geq 1$ such that $m_Y^d M_{YY}$ is a cancellative semigroup then $K[M]$ is right Noetherian.*
2. *the cancellative assumption holds for the semigroup $m_Y^d M_{YY}$ if and only if $a^k b^k = b^k a^k$ for all $a, b \in m_Y^d M_{YY}$ (for example if A is abelian).*
3. *$K[M]$ is right Noetherian if and only if $K[S]$ is right Noetherian, with $S = \{m \in M : \lambda_m = \mathrm{id}\}$ a submonoid of both $A(X, r)$ and $M(X, r)$.*
4. *if M also satisfies the ascending chain condition on left ideals (for example if r also is bijective) then $K[M]$ also is left Noetherian.*
5. *explicit formula for the Gelfand–Kirillov dimension of $K[M(X, r)]$ in terms of the number of orbits in X under actions of certain finite monoids derived from (X, r).*

Corollary (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

1. *If $K[A]$ is right Noetherian then so is $K[M]$.*
2. *If A is abelian then $K[M]$ is left and right Noetherian. Moreover, $M(X, r)$ is an epimorphic image of the Yang-Baxter monoid of a finite involutive solution.*
3. *If $K[M]$ is right Noetherian or semiprime, then M_{XX}^d is cancellative for some positive integer d. In particular, M_{XX}^d has a group of fractions that is abelian-by-finite.*
4. *If (X, r) is a finite left non-degenerate idempotent solution of YBE then*
    1. *$K[M]$ is left Noetherian and $\mathrm{GKdim} K[M] = 1$.*
    2. *$K[M]$ is right Noetherian if and only if the set $\Lambda = \{q(x) = \lambda_x^{-1}(x) : x \in X\}$ is a singleton.*
    3. *$K[M]$ is right Noetherian if and only if there exists a positive integer k such that $a^k b^k = b^k a^k$ for all $a, b \in M$.*

Problem 1

Let (X, r) be a finite left non-degenerate solution of the YBE. We know that $K[A(X, r)]$ is left Noetherian, but not necessarily right Noetherian.

Problem 1: Is $K[M(X, r)]$ always left Noetherian?

4. When is a Yang-Baxter algebra (semi)prime?

Theorem (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

The following conditions are equivalent for a finite left non-degenerate solution of YBE.

1. (X, r) *is an involutive solution.*
2. M *is a cancellative monoid and* Ω_λ *is a trivial group.*
3. $K[M]$ *is a prime algebra and* Ω_λ *is a trivial group.*
4. $K[M]$ *is a domain.*
5. $GKdim(K[M(X,r)]) = |X|$ *if and only if* (X, r) *is involutive.*

Moreover, if the diagonal map $q \colon X \to X : x \mapsto \lambda_x^{-1}(x)$*, is bijective then* Ω_λ *is a trivial group (for example if* r *is bijective).*

Let $v = eke_\sigma$ with $e = \exp(\mathrm{Im}(\lambda))$, $\sigma_x^{e_\sigma}$ idempotent for each $x \in X$, q^k idempotent and let $z = vx_1 + \cdots + vx_{|X|}$. Put $O = O_\lambda(eq(z_\kappa)) = \{\lambda_a(eq(z_\kappa)) : a \in A\}$. Then r_M restricts to an idempotent left non-degenerate solution on O. If M is cancellative then $\Omega_\lambda = \{(z, \lambda_z) \circ (eq(z_{\mathrm{id}}), \mathrm{id})^{-1} : z \in O_\lambda(eq(z_{\mathrm{id}}))\}$ is a finite subgroup of $(G(X,r), \circ)$, the group of fractions of M.

Problem 2

Problem 2: Do their exist prime algebras $K[M]$ with Ω_λ not trivial?

Theorem (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

Let (X, r) be a finite non-degenerate bijective solution of the YBE and let K be a field. The following properties are equivalent.

1. *$K[A]$ is a semiprime algebra.*
2. *A is a disjoint union*
$$A = \bigcup_{e \in \Gamma} A_e$$
of cancellative semigroups A_e indexed by a finite semilattice Γ such that
$$A_e A_f \subseteq A_{ef}$$
for all $e, f \in \Gamma$ (i.e., A is a finite semilattice Γ of cancellative semigroups) and each $K[A_e]$ is semiprime.

Moreover, in case the above equivalent conditions hold, Γ is the set of central idempotents of the classical ring of quotients of $K[A]$. Equivalently, A has a finite ideal chain with Rees factors cancellative semigroups that yield semiprime semigroup algebras. The latter condition holds in case K has zero characteristic.

Remark: If $Q = Q_{cl}(K[A])$ then $A_e = \{a \in A : Qa = Qe\}$ for $e \in \Gamma$.

Problem 3

Problem 3: When is K[M(X,r)] semiprime for (X, r) a finite left non-degenerate solution of YBE?

Determine Gelfand-Kirillov dimension of $K[M(X,r)]$

Theorem (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

Let (X,r) be a finite left non-degenerate solution of the YBE and let K be a field. Then

$$GKdimK[M(X,r)] = GKdimK[A(X,r)] = clKdimK[A(X,r)].$$

Moreover,

$$GKdimK[M(X,r)] = \max\{n_Y \mid Y \in \mathcal{Y}\}$$

where

$$\mathcal{Y} = \{\emptyset \neq Y \subseteq X : \sigma_y(Y) \subseteq Y \text{ and } \sigma_y(X \setminus Y) \subseteq X \setminus Y \text{ for all } y \in Y\}$$

and

$$n_Y = \text{ is the number of orbits of the set } Y \text{ with respect to the action of the monoid } \Sigma_Y = \langle \sigma_y : y \in Y \rangle.$$

In particular, all the above dimensions are bounded by $|X|$. Furthermore, if $K[M]$ is left or right Noetherian then also $GKdimK[M] = clKdimK[M]$.

The result is proven via a description of prime ideals of $K[A]$.

Corollary (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

Let (X, r) be a finite left non-degenerate solution of the YBE and let K be a field. Then

$$J(K[M(X,r)]) = B(K[M(X,r)]) = I(\eta_M) \cap \bigcap_{P \in \mathcal{P}} P,$$

where

$$\begin{aligned} \mathcal{P} &= \{P \in Spec(K[M]) : P \cap M(X,r) \neq \emptyset\} \\ &= \{P \in Spec(K[M(x,r)]) : M_{XX} \subseteq P\} \end{aligned}$$

and η_M is the cancellative congruence on M,

$$\begin{aligned} \eta_M &= \{(x,y) \in M \times M : x \circ z = y \circ \text{ for some } z \in M\} \\ &= \{(a, \lambda_a), (b, \lambda_b) \in M \times M : \lambda_a = \lambda_b \text{ and} \\ &\qquad a + iz_{\text{id}} = b + iz_{\text{id}} \text{ some } i \geq 1\} \end{aligned}$$

where $z_{\text{id}} = vx_1 + \cdots + vx_n$.

Theorem (Jespers, Kubat, Van Antwerpen 2019)

If (X, r) also is bijective and $K[M]$ is semiprime then there exist finitely many finitely generated abelian-by-finite groups $G_1 \cdots, G_k$, each being the group of quotients of a cancellative subsemigroup of M, such that $K[M]$ embeds into the direct product of matrix algebras $M_{n_1}(K[G_1]) \times \cdots \times M_{n_k}(K[G_k])$.

5. Characterisation of involutive solutions

Theorem (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

Let (X, r) be a finite left non-degenerate solution of the YBE and let If K be a field. The following conditions are equivalent:

1. (X, r) *is an involutive solution.*
2. $GKdimK[M] = |X|$.

Moreover, if $K[M]$ is left and right Noetherian then the above conditions are equivalent to:

3. $rkM = |X|$.
4. $clKdimK[M] = |X|$.
5. $\operatorname{id} K[M] = |X|$.
6. $K[M]$ *has finite global dimension.*
7. $K[M]$ *is Auslander–Gorenstein with* $\operatorname{id} K[M] = |X|$.
8. $K[M]$ *is Auslander-regular.*

5. Solutions that are degenerate

Theorem (Colazzo, Jespers, Kubat, Van Antwerpen 2023)

If $r(x,y) = (\lambda_x(y), \rho(x))$ is a solution (no other restrictions) then $K[M]$ is a right Noetherian PI-algebra of finite Gelfand–Kirillov dimension. If, furthermore, the solution (X, r) is left non-degenerate then $K[M]$ also is left Noetherian.

Problem

Let (X, r) be an arbitrary solution of the Yang-Baxter equation.

Problem 4: Determine when $K[(X, r)]$ is left/right Noetherian.

6. Some comments
6.1 Those who prefer groups: be careful

$G(X,r)$ the structure group, defined as a group by the "same" relations as $M(X,r)$.
X is embedded in $M(X,r)$ but $\iota : X \to G(X,r) : x \mapsto x$ is not necessarily injective. If it is then it is called an injective solution.

Theorem (Jespers, Kubat, Van Antwerpen, Vendramin 2021)

Let (X,r) be a finite non-degenerate solution of the YBE. Then the group $G(X,r)$ is torsion-free if and only if the injectivisation $(\iota(X), r_\iota))$ is an involutive solution.

Let G be a group and consider the map r on $G \times G$ defined as $r(x,y) = (y, y^{-1}xy)$ (Lebed and Vendramin) is an injective solution and if G is not abelian it is not involutive. So by the previous results $M(X,r)$ is not cancellative (as not involutive) and thus not embedded in $G(X,r)$.

6.2: Warning 1

For involutive non-degenerate solutions $K[M]$ shares many properties with commutative polynomial algebras. However, elementwise this is not the case.

Kaplansky's unit conjecture:
let K be a field and G a torsion-free group. Then the only invertible elements of the group algebra $K[G]$ are the obvious ones, i.e. elements of the form kg with $0 \neq k \in K$ and $g \in G$.

Obviously the conjecture holds if G is poly-infinite cyclic.

However, only in 2021 Gardam proved that for the group $P = \langle x, y \mid y^{-1}x^2y = x^{-2}\ x^{-1}y^2x = y^{-2} \rangle$ and K a field of $\mathrm{char}(K) = 2$ the conjecture has a negative answer. Murray 2021 extended this counter example to any field K of non-zero characteristic.

There is a structure group $G(X, r)$ of an involutive finite non-degenerate solution (X, r) that contains the group P. So, for such solutions of the YBE, $K[G(X, r)]$ can have invertible elements that are not obvious (at least in characteristic non-zero).

In this context it is interesting to determine when such $G(X, r)$ is poly-infinite cyclic.

Theorem (Bachiller, Cedó, Vendramin 2018)

Let (X, r) be a finite non-degenerate involutive set-theoretic solution of the Yang-Baxter equation. The following statements are equivalent:

1. (X, r) *is a multipermutation solution.*
2. $G(X, r)$ *is left orderable.*
3. $G(X, r)$ *is poly-infinite cyclic.*

Problem 5:
Has Kaplansky's unit conjecture a positive answer for group rings $R[G]$ of a torsion-free group G in case R is a field of zero characteristic or $R = \mathbb{Z}$?

6.3: Warning 2

Let (X, r) and (Y, s) be solutions of the Yang–Baxter equation. A map $f\colon X \to Y$ is a morphism of solutions if $(f \times f) \circ r = s \circ (f \times f)$. If f is bijective then (X, r) and (Y, s) are said to be isomorphic.
Two involutive non-degenerate solutions (X, r) and (Y, s) are isomorphic if and only if their structure monoids $M(X, r)$ and $M(Y, s)$ are isomorphic.
In general this is not true.

Example

Let $X = \{x_1, x_2, x_3\}$. Define $\sigma_1 = (1, 2)$, $\sigma_2 = (1, 3)$, $\sigma_3 = (1, 2)$ and consider the maps $r, s\colon X \times X \to X \times X$ given by

$$r(x_i, x_j) = (x_j, x_{\sigma_j(i)}) \qquad \text{and} \qquad s(x_i, x_j) = (x_{\sigma_i(j)}, x_i).$$

Both (X, r) and (X, s) are bijective ($r^3 = s^3 = \mathrm{id}$) non-degenerate solutions of the YBE and $M(X, r) = A(X, r) = A(X, s) = M(X, s)$. However, (X, r) and (X, s) are not isomorphic as solutions. Indeed, if $f\colon (X, r) \to (X, s)$ were an isomorphism of solutions then, in particular, $f \circ \sigma_x = f$ for all $x \in X$, which would lead to $\sigma_x = \mathrm{id}$, a contradiction.

7. Bijective non-degenerate finite solutions and the graded algebra $K[M(X,r)]$

(X,r) is a finite non-degenerate bijective solution of YBE.

$$M = \bigcup_m M_m$$

with

M_m the set of elements of M of length m.

Clearly,

$$K[A(X,r)] = \oplus_{m\geq 0} K[A]_m = \oplus_{m\geq 0} \mathrm{vect}_K(A_m)$$

$$K[M(X,r)] = \oplus_{m\geq 0} K[M]_m = \oplus_{m\geq 0} \mathrm{vect}_K(M_m)$$

are connected graded K-algebras with

$$\dim(K[A]_m) = \dim(K[M]_m) = |M_m| = |A_m|,$$

In particular, $\dim(A_2)$ is the number of r-orbits in X^2.

In the monoid $A = A(X, r)$: $Aa = aA$ for all $a \in A$. Hence

$$|M_2| = |A_2| \quad \leq \quad \text{the number of words of length 2 in the free abelian monoid of rank } |X| = n.$$

Thus

$$\dim A_2 \leq n + \binom{n}{2}.$$

If this upper bound is reached then this is called the maximality condition for $K[M(X, r)]$ and $M(X, r)$.

To determine $|M_2|$ it is sufficient to deal with the derived solution s, i.e. determine the number of s-orbits in X^2.

7.1 Maximality condition

Denote by $O_{(x,y)}$ the s-orbit of $(x,y) \in X^2$ We determine when this maximality condition holds.

Assume $\dim A_2 = n + \binom{n}{2}$. Then $n^2 = \sum_{i=1}^{n+\binom{n}{2}} |O_{(x_i,y_i)}|$, for some $(x_i, y_i) \in X^2$.
$s(x,y) = (y, \sigma_y(x)) \neq (x,y)$ if $x \neq y$ for x, y in X. Hence, for such elements $|O(x,y)| \geq 2$.
Thus, if there are m orbits with one element, then $n \geq m$ and

$$n^2 \geq m + 2(n + \binom{n}{2} - m) = n(n+1) - m = n^2 + n - m.$$

Hence, $n = m$ and thus $|O_{(x,x)}| = 1$ for all $x \in X$, and all other orbits have precisely 2 elements. Therefore s is involutive and each $\sigma_x = \mathrm{id}$, that is, $A(X,r)$ is the free abelian monoid of rank n. So, the maximality condition holds precisely when s, and thus r, is involutive and $A(X,r)$ is the free abelian monoid of rank $|X|$.

Theorem (Cedó, Jespers, Okninski 2021)

Let (X, r) be a finite non-degenerate bijective solution of the YBE and let K be a field. The following properties are equivalent:

1. $K[M(X, r)]$ *satisfies the maximality condition, that is* $\dim(K[M(X, r)])_2) = \binom{|X|}{2} + |X|$;
2. (X, r) *is involutive;*

Recall that the latter is further described in earlier stated results.

7.2 Minimality condition

Gateva-Ivanova introduced a "minimality condition" for a finite non-degenerate braided set (X, r), with a focus on square-free sets. She proposed to consider the case where dim A_2 is smallest possible.

Theorem (Cedó, Jespers, Okniński 2021)

Let K be a field and let (X, s) be a finite non-degenerate bijective solution of the YBE with all $\lambda_x = \mathrm{id}$ (so $r = s$). Then

$$\dim(K[A]_2) \geq \frac{|X|}{2}.$$

- *if $|X|$ is even then the lower bound $\frac{|X|}{2}$ is reached precisely when all σ_x, with $x \in X$, are equal to a cycle σ of length $|X|$.*
- *If $|X|$ is odd then the lower bound $\frac{|X|+1}{2}$ is reached when all σ_x, with $x \in X$, are equal to a cycle σ of length $|X|$.*

In particular, $s(a, b) = (b, \sigma(a))$ for all $a, b \in X$ and thus the solution (X, s) is an indecomposable multipermutation solution of level 1.

Example

Consider (X, r) with $X = \mathbb{Z}/(n)$, for an integer $n > 1$, and $r(x, y) = (y + 2, x - 1)$ for all $x, y \in X$. Then (X, r) is a non-degenerate bijective solution of YBE, its derived solution is (X, s), with $s(x, y) = (y, x + 1)$, for all $x, y \in X$. Then,

$$\dim(A(K, X, r)_m) = \dim(A(K, X, s)_m) = 1$$

for all $m > 2$, so that $GKdimK[A(X, r)] = GKdim(K[M(X, r)]) = 1$ for any field K. Furthermore

$$\dim K[M(X, r)]_2) = n/2 \text{ if } n \text{ is even,}$$

and

$$\dim(K[M(X, r)]_2) = (n + 1)/2 \text{ if } n \text{ is odd.}$$

7.3 Minimality condition for square free solutions

Theorem (Cedó, Jespers, Okniński 2021)

Let (X, r) be a finite bijective non-degenerate square free solution of YBE with $r = s$ (so $s(x,x) = (x,x)$). Then the number of r-orbits in X^2 is at least $2|X| - 1$, that is $\dim A_2 \geq 2|X| - 1$ where $A = A(K, X, r)$.

Theorem (Cedó, Jespers, Okniński 2021)

Let (X, r) be a finite non-degenerate bijective square-free solution of YBE with $r = s$. Suppose that $|X| > 1$ and that the number of r-orbits in X^2 is $2|X| - 1$. Then, up to isomorphism, one of the following holds

1. *$|X|$ is an odd prime and (X, r) is the braided set associated to the dihedral quandle,*
2. *$|X| = 2$ and (X, r) is the trivial braided set,*
3. *$X = \{1, 2, 3\}$ and $\sigma_1 = \sigma_2 = \mathrm{id}$, $\sigma_3 = (1, 2)$.*

An important step is to compute orbits.

Proposition (Cedó, Jespers, Okniński 2021)

Let (X, r) be a square-free non-degenerate bijective solution of the YBE with $r(x, y) = (y, \sigma_y(x))$ for allx, $y \in X$. Then,

$$r^{2k+1}(x, y) = ((\sigma_y \sigma_x)^k(y),\ (\sigma_y \sigma_x)^k \sigma_y(x))$$

and

$$r^{2k}(x, y) = ((\sigma_y \sigma_x)^{k-1} \sigma_y(x)), (\sigma_y \sigma_x)^k(y))$$

for all non-negative integers k.

8. How essential is the assumption of being a solution of YBE for structure on $K[M(X,r)]$?

Problem 6: How relevant is the YBE solution assumption in the obtained structural results for Yang-Baxter algebras?

Quadratic Monoids

Let X be a finite set of cardinality n and

$$r : X^2 \to X^2 : (x, y) \mapsto (\lambda_x(y), \rho_y(x))$$

a map (no other restrictions).

Definition

If r is involutive and r has precisely n fixed points (so $\binom{n}{2}$ non-fixed points),
i.e. $M(X, r)$ has a set of $\binom{n}{2}$ defining relations of the type $xy = uv$ with $(x, y) \neq (u, v)$,
then $M(X, r)$ is said to be a monoid of quadratic type.
In other words there are precisely n words of the type xy that are not rewritable.
If it is square free, i.e. $r(x, x) = (x, x)$ then $M(X, r)$ is said to be of skew type.

If r is left non-degenerate then for each $x \in X$ there exists a unique $y \in X$ such that xy is not rewritable.
Again via left divisibility one proves the following result, where M_i denotes the elements of M left divisible by at most i elements of X.

Theorem (Jespers, Okniński, Van Campenhout 2015)

If $M(X,r)$ is a left non-degenerate quadratic monoid. Then,

$$M(X,r)\setminus\{1\}=\{w_1\cdots w_q \mid 1\leq q\leq n,\ w_i\in A_k \text{ for some } 1\leq k\leq n\}$$

where the set of all non-rewritable elements of is

$$M_1\setminus M_2 = A_1\cup\cdots\cup A_n$$

and each A_i consists of subwords of an infinite periodic word of period not exceeding n, i.e. words of the type

$$x_{i_1}\cdots x_{i_{k-1}}(x_{i_k}\cdots x_{i_r})^t x_{i_k}\cdots x_{i_p}$$

In particular the Gelfand-Kirillov dimension does not exceed n. Also, there exists a positive integer l so that M_n^l is a cancellative semigroup. If, furthermore (X,r) is non-degenerate, then $K[M(X,r)]$ is left and right Noetherian and satisfies a polynomial identity.

This was proven earlier by Cedó and Okninski for non-degenerate monoids of skew type.

To prove this we also make use of the following result

Theorem (Gateva–Ivanova, E. Jespers and J. Okniński 2003)

Let S be a monoid such that the algebra $K[S]$ is right Noetherian and $GKdim(K[S]) < \infty$. If S is finitely generated and if S has a monoid presentation of the form

$$S = \langle x_1, \ldots, x_n \mid R \rangle$$

where R is a set of homogeneous relations (i.e. every relation of the form $u = v$ with u, v words of the same length in the free monoid on X), then $K[S]$ satisfies a polynomial identity.

But we can do better.

Theorem (Jespers, Van Campenhout 2017)

Let $M(X,r)$ be a finitely generated non-degenerate quadratic monoid. Then there exists a positive integer N so that

$$A = \langle s^N \mid s \in M \rangle \text{ is abelian}$$

and

$$M = \bigcup_{f \in F} fA = \bigcup_{f \in F} Af$$

for some finite subset F of M.

Theorem (Jespers, Van Campenhout 2017)

Let $M(X,r)$ be a finitely generated non-degenerate monoid of quadratic type. Then the following properties are equivalent:

1. *(X,r) is a set-theoretic solution of the YBE,*
2. *$M(X,r)$ is cancellative and satisfies the cyclic condition, that is, if $x_1, y, y_1, z_1 \in X$ and $r(x_1, y) = (y_1, z_1)$ then $r(x_2, y_1) = (y_2, z_2)$ for some $x_2, y_2, z_2 \in X$ with x_2x_1 and z_2z_1 non-rewritable.*

For skew type, i.e. square free solutions we have $x_2 = x_1$ and $z_2 = z_1$. Hence cyclic means:

$$\text{if } r(x_1, y) = (y_1, z_1) \text{ then } r(x_1, y_1) = (y_2, z_1) \text{ for some } y_2$$

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