Germs and Sylows for structure group of solutions to the Yang–Baxter equation

YRAC23 Conference

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26th July 2023, L'Aquila

Yang–Baxter Equation

Set-theoretical solution of the YBE *(Drinfeld '92)*

(X, r) where X is a set and $r : X \times X \to X \times X$ a bijection, such that

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

where $r_i : X \times X \times X \to X \times X \times X$ acts on the coordinates i and $i + 1$.

For any X, $r(x, y) = (y, x)$ defines a solution.

Definition *(Etingof–Schedler–Soloviev '99)*

Denote $r(x, y) = (\lambda(x, y), \rho(x, y))$. (X, r) is said to be:

- Involutive if $r^2 = \mathrm{id}_{X \times X}$
- Left non-degenerate (resp. right) if $\lambda(x, -)$ (resp. $\rho(-, y)$) is a bijection for any x (resp. y).

$\Rightarrow$ X is determined by the permutations $\lambda(x, -)$, $x \in X$.

Structure groups

Definition *(Etingof–Schedler–Soloviev '99)*

Define the structure group G (resp. monoid M) by the presentation

$$\langle X \mid xy = x'y' \text{ if } r(x,y) = (x',y') \rangle$$

$X = \{x_1, x_2\}$ with $\lambda(x_i, -) = (12)$ yields $M = \langle x_1, x_2 \mid x_1^2 = x_2^2 \rangle^+$.

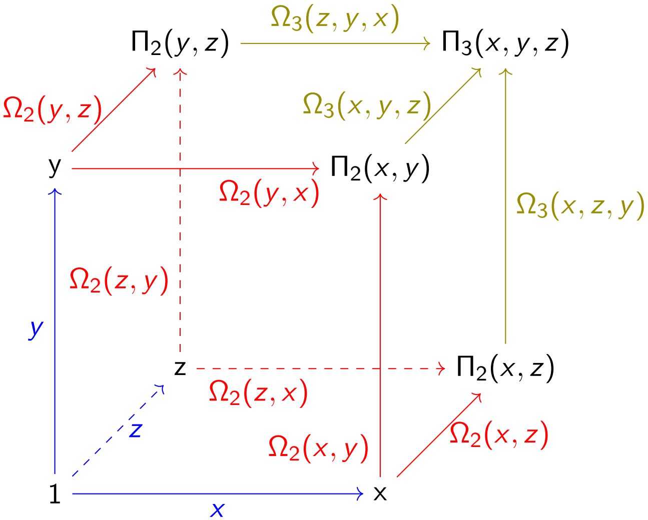
From now on we suppose X to be finite.

Theorem *(Chouraqui '10)*

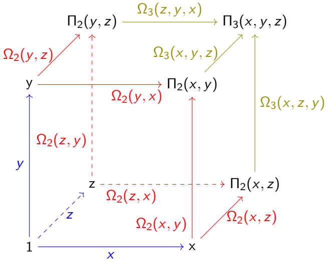
G is a Garside group.

• Garside $\Rightarrow$ solutions to the word and conjugacy problems, torsion-free, normal forms...

Cube condition



Garsideness



- Distance from 0 gives a length ($l(ab) \geq l(a) + l(b)$)
- M is cancellative ($f_1 g f_2 = f_1 h f_2 \Rightarrow g = h$)
- M admits right gcd and lcm (meet and join, min and max)
- M admits left gcd and lcm
- The left and right lcm of the generators coincide (Δ), so does its left and right divisors set

$\Rightarrow M \hookrightarrow G = \mathrm{Frac}(M)$.

Dehornoy's class and germ

$$1 \xrightarrow[\Omega_1(x)]{} x \xrightarrow[\Omega_2(x,x)]{} x^{[2]} \dashrightarrow x^{[k-1]} \xrightarrow[\Omega_k(x,x,\dots,x)]{} x^{[k]}$$

Proposition (Dehornoy's class)

There exists $d \in \mathbb{N}$ such that $\Omega_{d+1}(x, \dots, x, y) = y$ for all $x, y \in X$.

Example: If $X = \{x_1, \dots, x_n\}$ with $\lambda(x_i, -) = (12 \dots n)$, $d = n$ and $x_1^{[n]} = x_1 x_2 \dots x_n$.

Theorem (Germ) (Dehornoy '15)

(M, Δ^{d-1}) can be "recovered" from $\overline{G} = G/\langle x^{[d]} \rangle$ which is finite.

- The Cayley graph of G is determined by the finite subgraph of the cube with side length d.

Sylow for the germs

Theorem *(Lebed-Ramírez-Vendramin '22, F.)*

$G^{[k]} = \langle x^{[k]} \rangle$ induces the structure of a solution on $X^{[k]} = \{x^{[k]}\}_{s \in S}$.
Moreover, its class is $\frac{d}{d \wedge k}$ (if $k \leq d$).

- The subgraph "generated" by the $x^{[d]}$ is the Cayley graph of a structure group.
- Decompose $d = p_1^{a_1} \dots p_r^{a_r}$, and let $\alpha_i = p_i^{a_i}, \beta_i = \frac{d}{\alpha_i}$.

Lemma

The $\overline{G}^{[\beta_i]}$ are p_i-Sylow of $\overline{G}$, they commute two by two and their product is $\overline{G}$.

($H, K < G$ commute means $HK = KH$, i.e $\forall h, k, \exists h', k', hk = k'h'$.)

- We have an "easy" way to reverse this process : constructing new solutions from ones with same size (Cf. Matched Product).

Questions

1. What can we say about the possible values and bounds on the class d?
2. When is the representation irreducible? What does it mean?
3. Can we find other Garside elements to change germs?
4. Can Hecke algebras be defined for our germs?
5. How does the Garside approach generalizes to Weyl groups? To the degenerate case?

Thank you for your attention!