Labelling Hasse diagrams of modular geometric lattices

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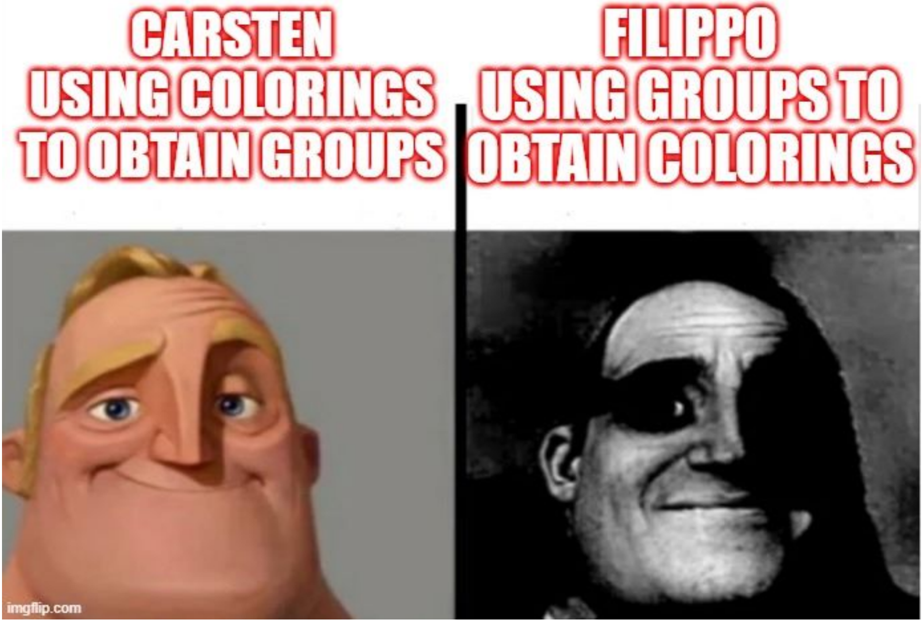
L'Aquila - July 28th, 2023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Definition: Join / Meet

Let L be a partially ordered set and $x, y, z \in L$.

- Write $z = x \vee y$, if

$$\forall a \in L : \ (\,(a \geq x) \,\&\, (a \geq y)\,) \ \Leftrightarrow (a \geq z).$$

  We say, z is the join of x, y.
- Write $z = x \wedge y$, if

$$\forall a \in L : \ (\,(a \leq x) \,\&\, (a \leq y)\,) \ \Leftrightarrow (a \leq z).$$

  We say, z is the meet of x, y.

Definition: Lattice

A partially ordered set L is a lattice if all binary joins and meets exist.

- Partially ordered sets do not admit an algebraization.
- But lattices do!

Identities for meets and joins

Lattices form a variety of algebras of type $(2, 2)$:

$$\begin{aligned} x \wedge x = x \quad &, \quad x \vee x = x\,, \\ (x \wedge y) \wedge z = x \wedge (y \wedge z) \quad &, \quad (x \vee y) \vee z = x \vee (y \vee z)\,, \\ x \wedge y = y \wedge x \quad &, \quad x \vee y = y \vee x\,, \\ (x \wedge y) \vee x = x \quad &, \quad (x \vee y) \wedge x = x. \end{aligned}$$

You get more specific varieties by adding further identities:

The modular variety

Modular lattices are defined by the property

$$x \leq z \Rightarrow (x \vee y) \wedge z = x \vee (y \wedge z).$$

This can be rewritten as the identity

$$(x \vee y) \wedge (x \vee z) = x \vee (x \wedge (x \vee z)).$$

Modular geometric lattices

Definition: Bounded lattice / Atom / Geometric lattice

Let L be a lattice.

- L is bounded if it has a maximal element 1_L and a minimal element 0_L.
- If L is bounded, an element $x \in L$ is an atom if $x \succ 0$.
  Here, $x \succ y$ means: $x > y$ and there is no z with $x > z > y$.
  Write $\mathrm{Atom}(L)$ for the set of atoms in L.
- If L is modular and bounded, we call L geometric if L is also atomistic, meaning that
$$L = \left\{ \bigvee A \ :\ A \subseteq \mathrm{Atom}(L),\ 0 < |A| < \infty \right\}.$$

Some bounded modular geometric lattices

- The power set $\mathcal{P}(X)$ of a finite set X under inclusion.
- $L(K, n) = \{K\text{-subspaces of } K^n\}$ under inclusion.
- Subspace lattices of non-desarguesian planes.
- Degenerate geometries: The geometry of points on a line.
- Products of all of the above!

Rump's labelling procedure - the rules

If P is a partially ordered set, the Hasse diagram of P is the set

$$\mathrm{Has}_P = \left\{(x, y) \in P^2 : x \prec y\right\}.$$

For a set X, write $X^{(2)} = \{(x, y) \in X^2 : x \neq y\}$.

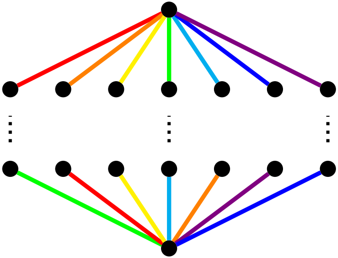
Rump's Labelling Procedure

Let L be a bounded modular geometric lattice. A non-degenerate block labelling of Has_L is given by a set of colours Col and a map $\chi : \mathrm{Has}_L \to \mathrm{Col}$ such that:

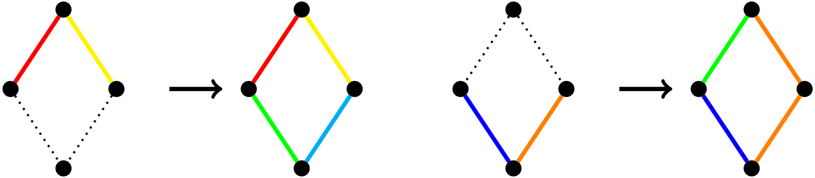
- χ restricts to a bijection $\{(0_L, x) : x \succ 0_L\} \to \mathrm{Col}$.
- χ restricts to a bijection $\{(x, 1_L) : x \prec 1_L\} \to \mathrm{Col}$.
- There is a bijection $D : \mathrm{Col}^{(2)} \to \mathrm{Col}^{(2)}$ such that for any two chains $a \prec b \prec d$, $a \prec c \prec d$ ($b \neq c$), there are w, x, y, z such that
  - $(w, x) \in \mathrm{Col}^{(2)}$,
  - $(y, z) = D(w, x)$,
  - $\chi(a, b) = w$, $\chi(a, c) = x$, $\chi(b, d) = y$, $\chi(c, d) = z$.

A visualization of Rump's labelling rules

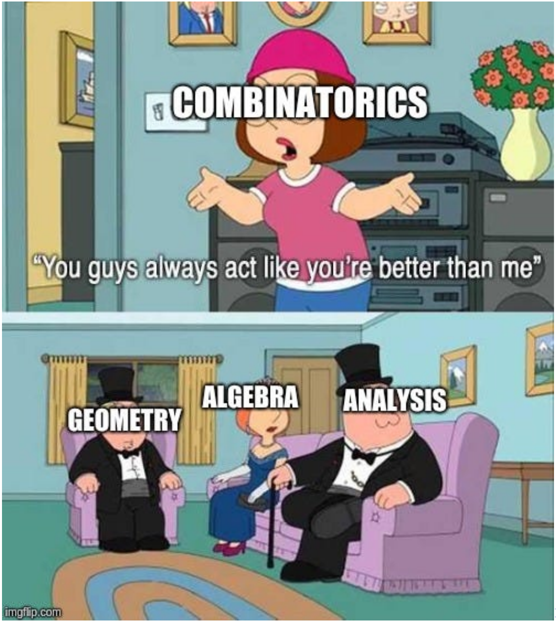
- Each colour is present exactly once in the upper and lower layer:



- Given the colours of the upper or lower two edges of a quadrangle, there is a unique way to colour the other ones:



What are these colourings good for?



Rump's method of constructing modular quasi-Garside groups

The group of a block labelling

Let L be a bounded modular geometric lattice and $\chi : \mathrm{Has}_L \to \mathrm{Col}$ a non-degenerate block labelling thereof. Then χ defines the group

$$G_\chi = \left\langle \ \mathrm{Col} \ \middle| \ \begin{array}{c} \chi(a,b)\chi(b,d) = \chi(a,c)\chi(c,d) \\ \text{when } a \prec b \prec d;\ a \prec c \prec d \end{array} \right\rangle .$$

Theorem (Rump, 2015)

Let χ be a non-degenerate block labelling of a bounded modular geometric lattice.

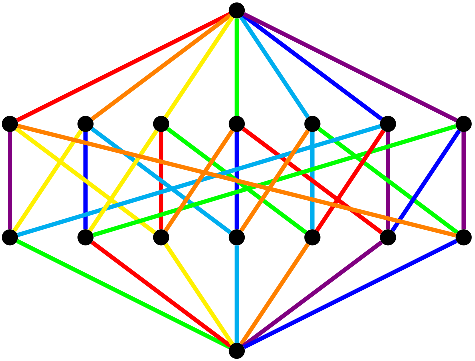
- G_χ is a group of fractions for the submonoid $M_\chi \subseteq G_\chi$ generated by Col.
- G_χ is a quasi-Garside group with $\Delta = \chi(a_0, a_1)\chi(a_1, a_2)\ldots\chi(a_{k-1}a_k)$ for some chain $0_L = a_0 \prec a_1 \prec \ldots \prec a_{k-1} \prec a_k = 1_L$.
- With respect to left divisibility, there is a lattice isomorphism

$$\{\, a \in M_\chi \ : \exists x \in M_\chi : ax = \Delta\} \cong L.$$

- G_χ is a modular lattice under left- (resp. right-) divisibility with respect to M_χ.

Each modular quasi-Garside group is of the form G_χ for some labelling χ of a bounded modular geometric lattice.

$L = L(\mathbb{F}_2, 3)$



Generators:

$\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$

Relations:

$\chi_7\chi_1 = \chi_3\chi_2 = \chi_5\chi_6$
$\chi_6\chi_2 = \chi_3\chi_3 = \chi_4\chi_7$
$\chi_3\chi_1 = \chi_1\chi_3 = \chi_2\chi_4$
$\chi_5\chi_2 = \chi_6\chi_4 = \chi_2\chi_5$
$\chi_4\chi_3 = \chi_5\chi_5 = \chi_1\chi_6$
$\chi_1\chi_4 = \chi_7\chi_6 = \chi_6\chi_7$
$\chi_2\chi_1 = \chi_4\chi_5 = \chi_7\chi_7$

Garside element:

$\Delta = \chi_4\chi_7\chi_1$

Constructing labellings by orthogonality

Let $L = L(\mathbb{R}, n)$.

- Take $\mathrm{Col} = \{\, \mathbb{R}v \;:\; v \in \mathbb{R}^n \setminus \{0\} \,\}$ as the set of colours.
- For $U \in L(\mathbb{R}, n)$, let $U^\perp = \{v \in \mathbb{R}^n \;:\; \forall u \in U : \langle u|v\rangle = 0\}$.
- For subspaces $U \prec V \leq L(\mathbb{R}, n)$, choose

$$\chi(U, V) = V \cap U^\perp \in \mathrm{Col}.$$

Theorem (Dietzel, 2019)

- χ is a non-denegerate block labelling.
- The resulting quasi-Garside group G_χ is the pure paraunitary group

$$\mathrm{PPU}_n = \{\, M(t) \in \mathbb{R}[t, t^{-1}]^{n\times n} \;:\; \underbrace{M(t^{-1})M(t)^\top = E_n}_{\text{paraunitarity}},\ \underbrace{M(1) = E_n}_{\text{purity}} \,\}.$$

- An isomorphism is given by the assignment $[v] \mapsto \left(E_n - \frac{|v\rangle\langle v|}{\langle v|v\rangle}\right) t + \frac{|v\rangle\langle v|}{\langle v|v\rangle}$.

An infinitude of finite examples - $L(\mathbb{F}_q, n)$

Constructing labellings using finite fields

Let $L = L({}_{\mathbb{F}_q}\mathbb{F}_{q^n}) \cong L(\mathbb{F}_q, n)$.

- Take $\mathrm{Col} = \{\, \mathbb{F}_q v \;:\; v \in \mathbb{F}_{q^n} \setminus \{0\} \,\}$ as the set of colours.
- For $U \in L({}_{\mathbb{F}_q}\mathbb{F}_{q^n})$, let

$$p_U(x) = \prod_{u \in U} (x - u).$$

  This is an $\mathbb{F}_q$-linear endomorphism of $\mathbb{F}_{q^n}$.
- For $\mathbb{F}_q$-subspaces $U \prec V \leq \mathbb{F}_{q^n}$, let

$$\chi(U, V) = p_U(V) \in \mathrm{Col}.$$

Theorem (Dietzel, 2021)

- χ is a non-denegerate block labelling.
- Let $\mathbb{F}_{q^n}[x, \sigma]$ be the twisted polynomial ring with respect to $\sigma(k) = k^q$ and $\mathbb{F}_{q^n}(x, \sigma)$ its quotient field.
  The assignment $\mathbb{F}_q v \mapsto x^q - v^{q-1}x$ embeds G_χ as a subgroup of $\mathbb{F}_{q^n}(x, \sigma)$.
- This construction can be generalized to cyclic field extensions!

Comments and problems

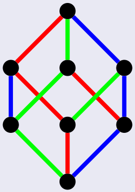
What about Boolean lattices?

Non-degenerate labellings of $\mathcal{P}(X)$ are equivalent to non-degenerate, involutive set-theoretic solutions of the Yang-Baxter equation on X.

Example:

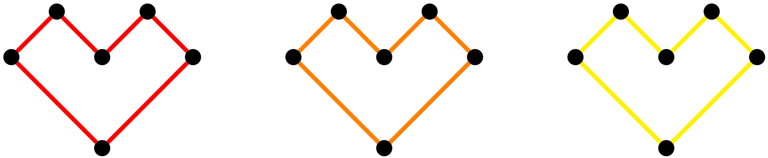
$$X = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$$

$$r(x, y) = (y + 1, x - 1)$$



Problems

- Does every desarguesian lattice $L(K, n)$ admit a non-degenerate block-labelling?
- Can lattices of non-desarguesian planes admit non-degenerate block labellings?
- Describe G_χ if χ is a block labelling derived from *one-sided* orthogonality, i.e. with respect to a non-hermitean, anisotropic sesquilinear form.



Thanks for your attention!