Component group of the centralizer of a nilpotent orbit

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A Lie algebra $\mathfrak{g}$ is a vector space together with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ mapping $(x, y) \to [x, y]$, such that:

1. it is bilinear;
2. $[x, x] = 0$ for all $x \in \mathfrak{g}$;
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Definition

A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Definition

A Lie algebra is simple if it is non-abelian and it has no non-trivial ideals.

$$\begin{aligned} \mathrm{ad}\colon\ \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ x &\to (y \to [x, y]) \end{aligned}$$

Root space decomposition of (semi)simple $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

- $\mathfrak{t}$ is a maximal toral subalgebra (i.e. it consists of elements $x \in \mathfrak{g}$ such that $ad(x)$ is diagonalizable);
- $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid (\mathrm{ad}\, t)x = \alpha(t)x \text{ for all } t \in \mathfrak{t}\}$, for $\alpha \in \mathfrak{t}^*$;
- $\Phi = \{\alpha \in \mathfrak{t}^* \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\}$.

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- A base π is a subset of Φ such that $\text{span}(\pi) = \mathfrak{t}^*$ and each $\beta \in \Phi$ can be written as $\sum_{\alpha \in \pi} k_\alpha \alpha$ for integer coefficients k_α all non-positive or all non-negative.
- Given a base $\pi = \{\alpha_1, \ldots, \alpha_n\}$, $\pi' = \pi \cup \{\alpha_0\}$, where $\alpha_0 = -\sum_{i=1}^n k_i \alpha_i \in \Phi$ such that $\sum_{i=1}^n k_i$ is maximal.

Definition

For a subset $J \subset \pi$ (resp. $J \subset \pi'$) and $\Phi_J \subset \Phi$, a Levi (resp. pseudo-Levi) subalgebra is a subalgebra of $\mathfrak{g}$ of the form:

$$\mathfrak{g}_J = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha.$$

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Complex algebraic group: G is an algebraic variety over $\mathbb{C}$ endowed with a group structure such that the operations $\mu : G \times G \to G$, $\mu(x, y) = xy$, and $\iota : G \to G$, $\iota(x) = x^{-1}$ are regular maps.

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Simple: G is non-abelian and has no nontrivial closed connected normal subgroups.

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Adjoint type: $G \cong AdG = \mathrm{Aut}(\mathfrak{g})^0$, i.e. the image of G by its adjoint representation $\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$ mapping $g \to d(y \to xyx^{-1})_e$.

Definition

An element $N \in \mathfrak{g}$ is said to be nilpotent if adN is a nilpotent endomorphism.

Definition

For a nilpotent element $N \in \mathfrak{g}$, we can define a nilpotent orbit as $O_N := \{\phi(N) \mid \phi \in \text{Aut}(\mathfrak{g})^0\}$.

Definition

A nilpotent element $N \in \mathfrak{g}$ is said to be distinguished if the only Levi subalgebra of $\mathfrak{g}$ containing N is $\mathfrak{g}$ itself.
If N is a distinguished nilpotent element, its orbit is also said to be distinguished.

Theorem (Bala-Carter, 1976)

There is a bijection between nilpotent orbits in $\mathfrak{g}$ *and pairs* $(\mathfrak{l}, N)$*, where* $\mathfrak{l}$ *is a Levi subalgebra of* $\mathfrak{g}$ *and* N *is a distinguished nilpotent element in* $\mathfrak{l}$*.*

Consequence: Classification of nilpotent orbits for simple Lie algebras.

Definition

We define the centralizer of N in G as $Z_G(N) := \{g \in G \mid (Adg)N = N\}$ and the component group of N as $A(N) := Z_G(N)/Z_G^0(N)$.

Theorem (Sommers, 1998)

There is a bijection between pairs (N, C) where N is a nilpotent element in $\mathfrak{g}$ and C a conjugacy class in $A(N)$, and pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a pseudo-Levi subalgebra of $\mathfrak{g}$ and N is a distinguished nilpotent element in $\mathfrak{l}$, up to G-conjugation.

Consequence: Determination of $A(N)$. It turns out that, for nilpotent elements of simple Lie algebras, $A(N)$ is isomorphic to one of S_n for $1 \leq n \leq 5$.

Purpose of my current work: Explicit computation of $A(N)$ for all simple Lie algebras.

Fix $J \subset \pi'$ and let $\mathfrak{g}_J =$ Lie(G_J) be the corresponding pseudo-Levi subalgebra. Fix a distinguished nilpotent element $N \in \mathfrak{g}_J$.

Facts

- *$Z(G_J)/Z(G_J)^0$ is a finite cyclic group, generated by a certain $x \in G_J$.*
- *$(Adx)N = N$, hence $x \in A(N)$.*
- *For any $N' \in \mathfrak{g}$ distinguished nilpotent and $g \in G$ such that $(Adg)N = N'$, $gxg^{-1} \in A(N')$.*

Algorithm for the computation of the component group $A(N)$ (implemented on GAP):

- Fix $J, J' \in \pi'$;
- List nilpotent orbits of g_J and g'_J;
- List distinguished nilpotent orbits of g_J and g'_J;
- Fix a distinguished nilpotent element N for g_J and N' for g'_J;
- Find x, x' which generate $Z(G_J)/Z(G_J)^0$ and $Z(G'_J)/Z(G'_J)^0$, respectively, having $x \in A(N)$ and $x' \in A(N')$.
- Find $g \in G$ such that $(Adg)N' = N$.
- Conclusion: $A(N) =< x, gxg^{-1} >$.

Key fact: We can use Sommers' correspondence to make a proper choice of J, J'.

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