On the Non-Transitivity Property of Group Actions in Cryptography

Giuseppe D'Alconzo*

Department of Mathematical Sciences,
Polytechnic of Turin

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*joint work with A. Flamini and A. Gangemi

Group actions in Cryptography

Let X be a set, G be a group and $\star : G \times X \to X$.
$(G, X, \star)$ is a *group action* if $\star$ is compatible with the group operation:

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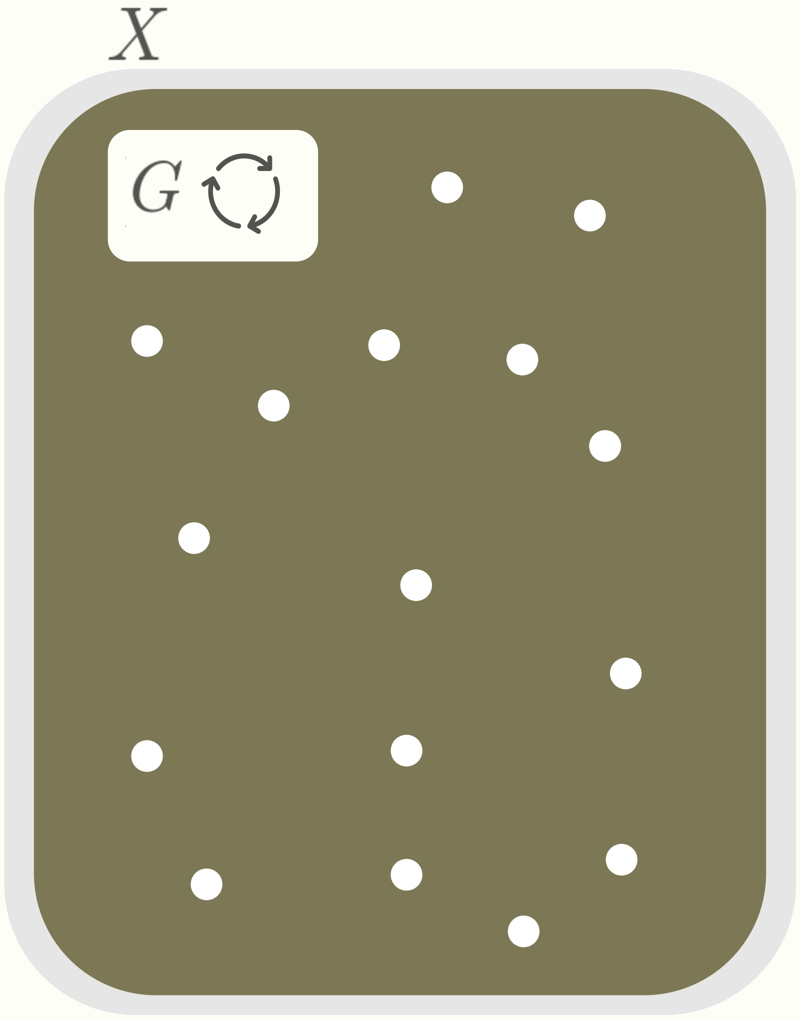
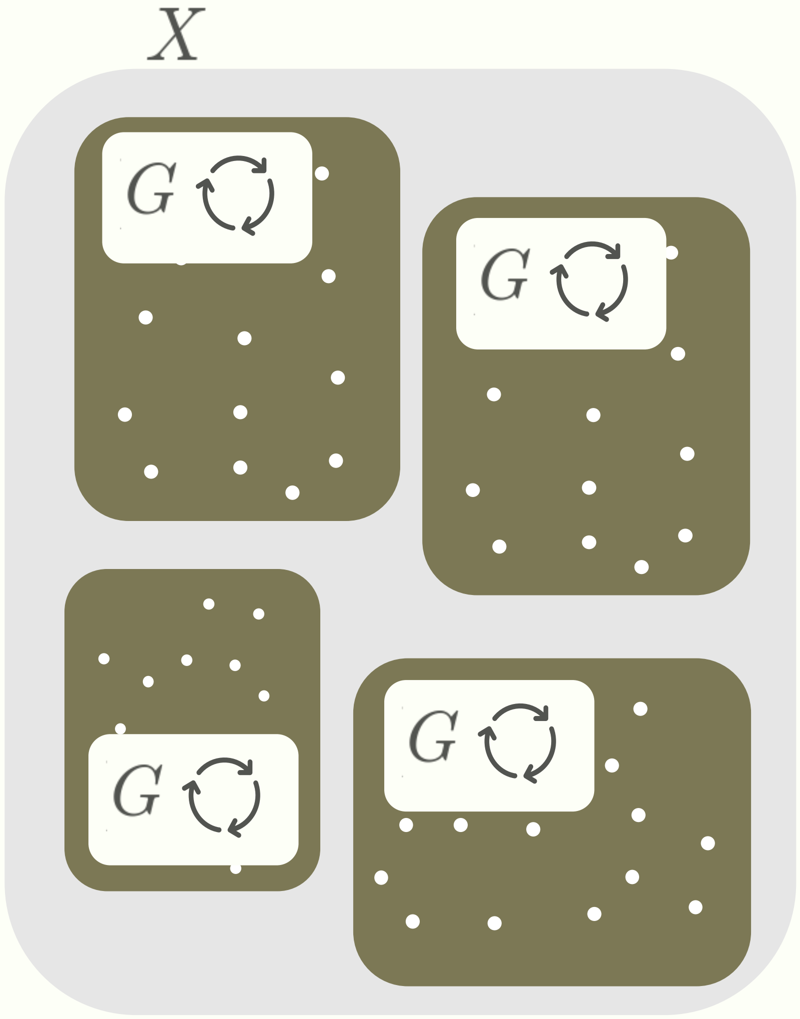
Many constructions from GAs!
Key exchanges, digital signatures, oblivious transfers, PRFs, etc.

Alamati, De Feo, Montgomery, Patranabis. "Cryptographic group actions and applications." Asiacrypt 2020.

On the Transitive Property





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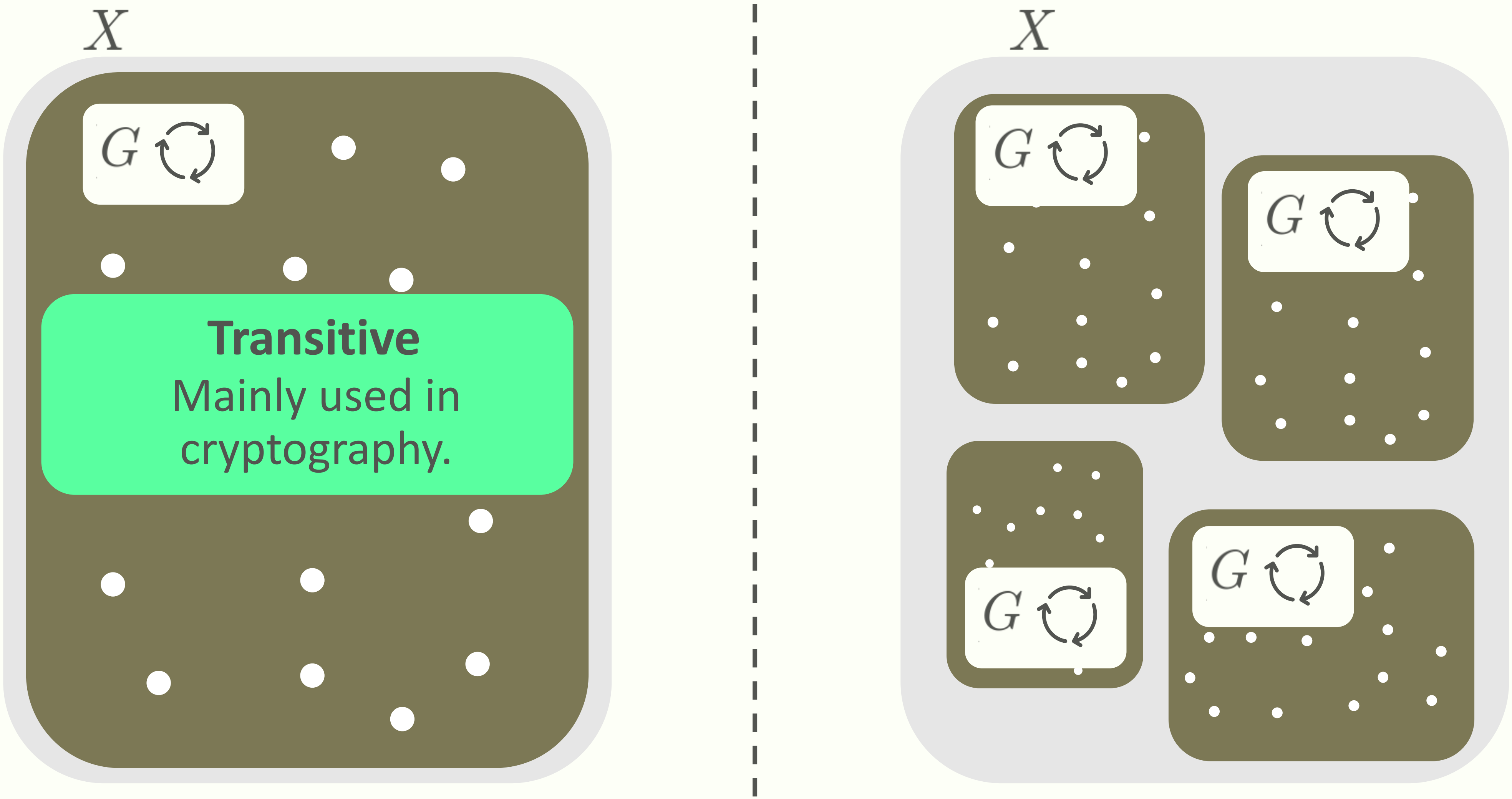

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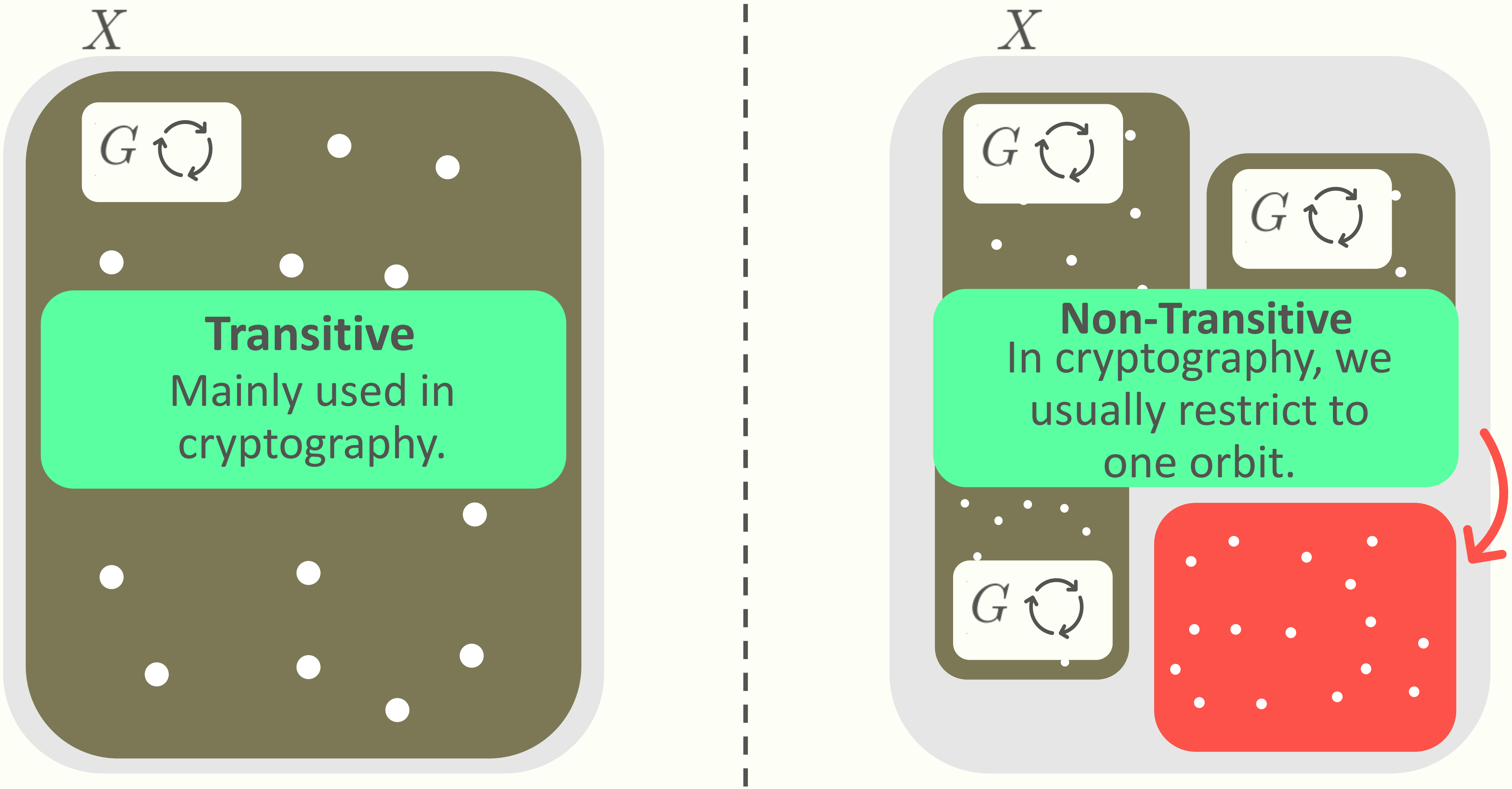
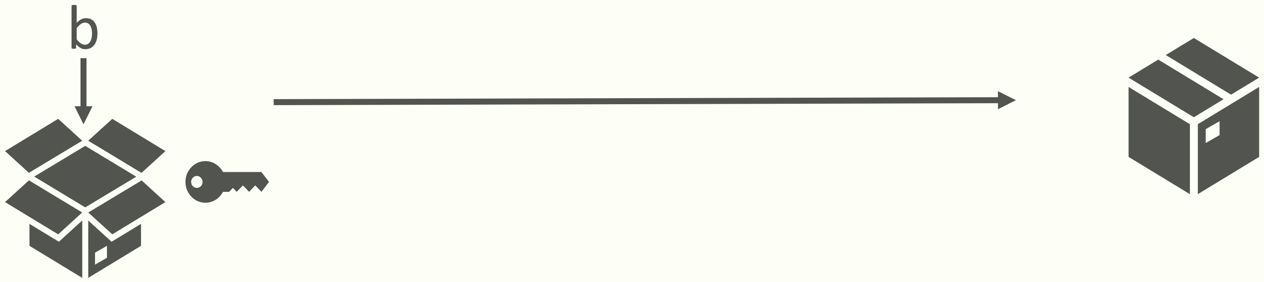


Commitment Scheme

Commitment Scheme

Commit phase:



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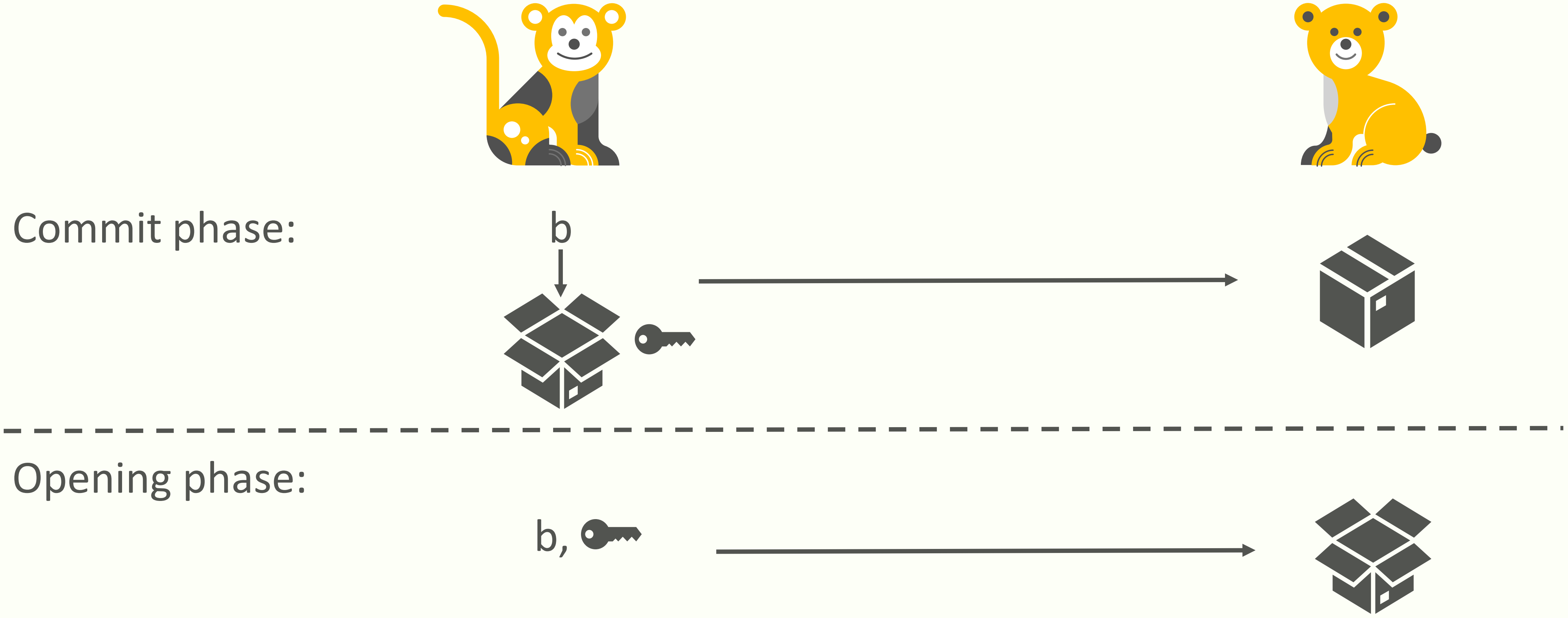

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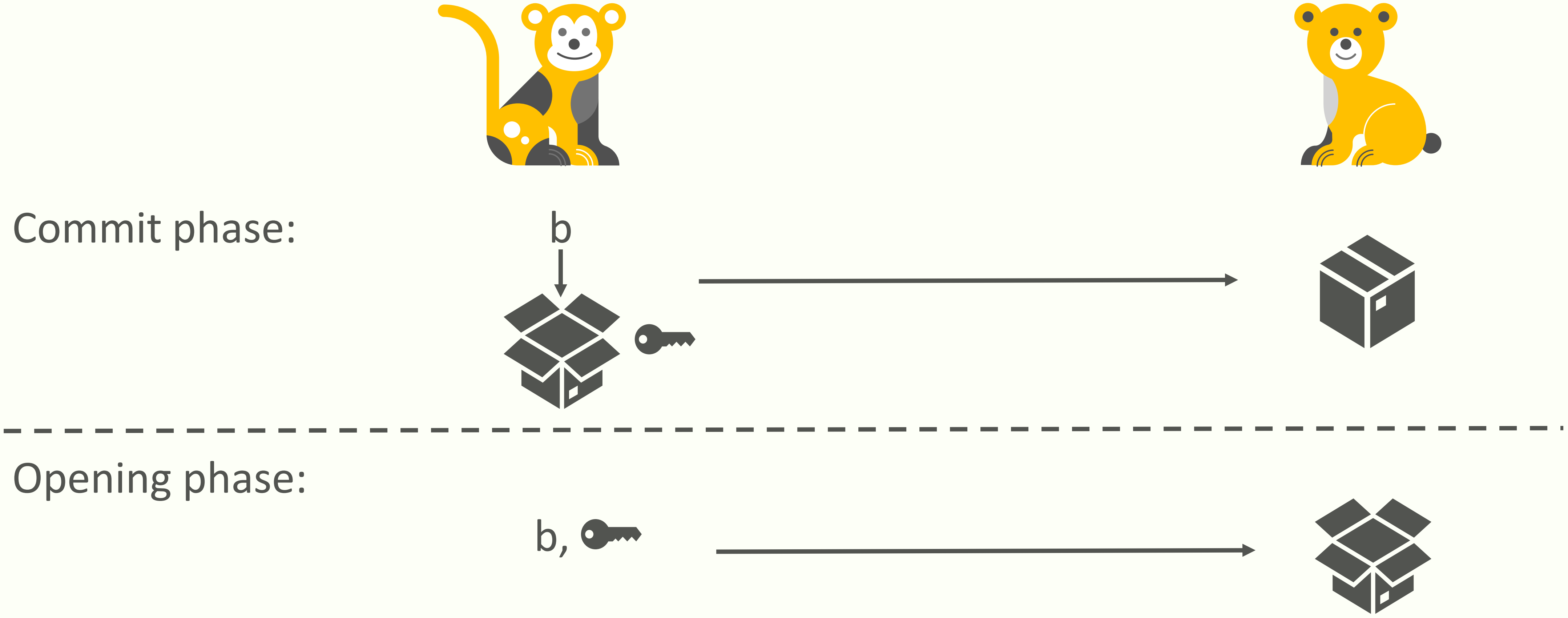

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The sender cannot change the value b.

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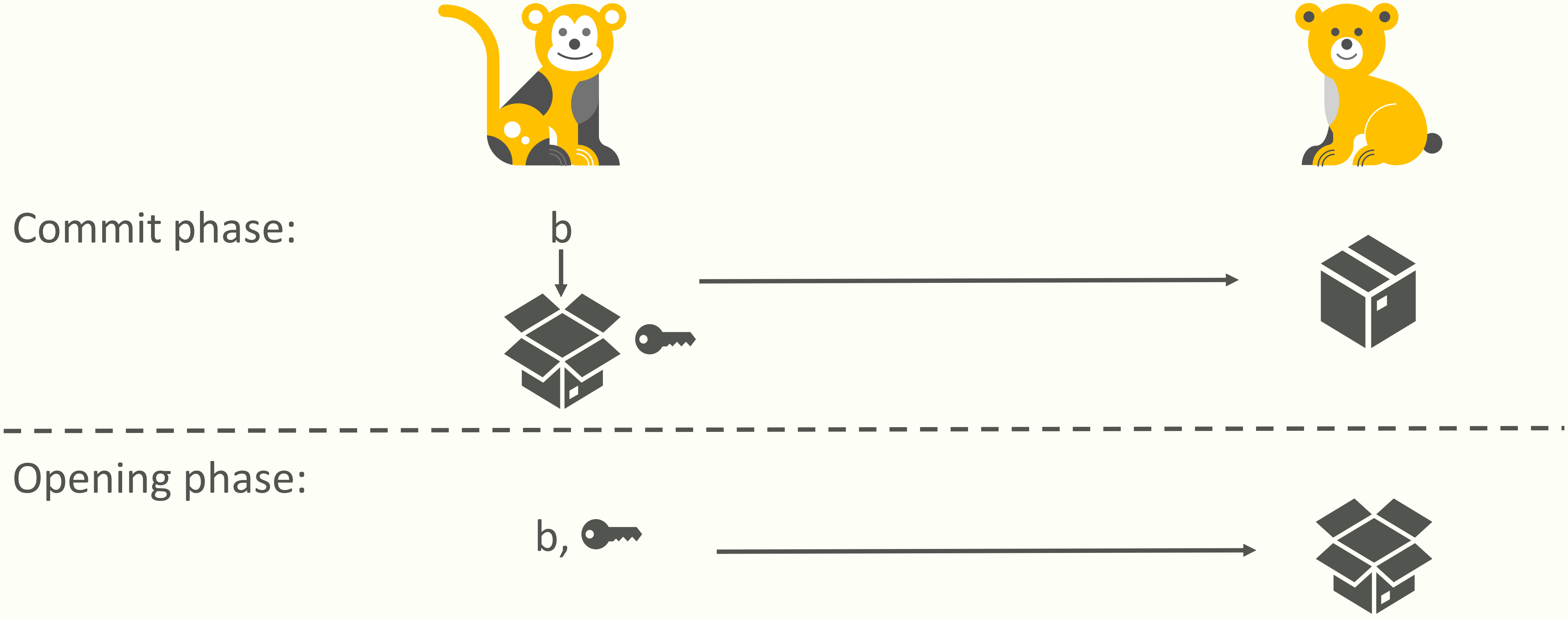

Binding
The sender cannot change the value b.

Hiding
does not give information on b.

Bit Commitment from GAs [BY91, JQSY19]

b

Public parameters
$(G, X, \star)$

Brassard, Yung. "One-way group actions." *CRYPTO 1990.*
Ji, Qiao, Song, Yun. "General linear group action on tensors: A candidate for post-quantum cryptography." *TCC* 2019.

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Setup phase: Checks π $\xleftarrow{x_0, x_1, \pi}$ $g, x_0, x_1 = g \star x_0$
proof π

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Issue 1
π can be huge

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Issue 2
Interaction!

Opening phase:

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A first attempt

Public parameters
$$(G, X, \star)$$

b

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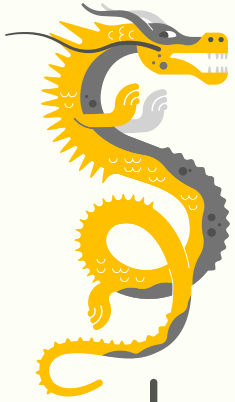

Trusted
third party
(TTP)

x_0, x_1 in
different orbits.

b

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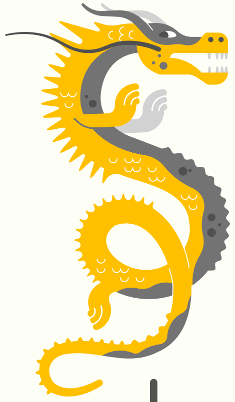

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Commit phase:

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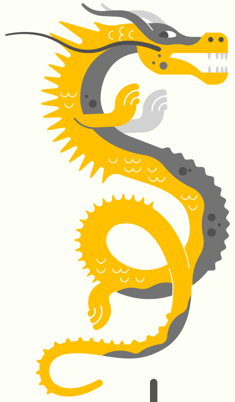

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$\xrightarrow{\text{b}, g}$ Checks $c = g \star x_b$

Removing the TTP

First try

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 generates x_0, x_1 in different orbits.

 b

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Removing the TTP

First try

Public parameters
$(G, X, \star)$

generates x_0, x_1 in different orbits.

Issue

can cheat by choosing $x_1 = h \star x_0$. Then he can open a different bit from b.

b

Commit phase:

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$c = g \star x_b$ $\xrightarrow{x_0, x_1, c}$ c

Opening phase:

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Removing the TTP

Second try

Public parameters
$(G, X, \star)$

 generates x_0, x_1 in different orbits and a proof π of that.

 b

Commit phase:

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c

Opening phase:

$\xrightarrow{\text{b}, g}$ Checks
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Removing the TTP

Second try

Public parameters
$(G, X, \star)$

 generates x_0, x_1 in different orbits and a proof π of that.

Issue

No suitable techniques to produce such a non-interactive proof π.

 b

Commit phase:

g

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c

Opening phase: $\xrightarrow{\text{b}, g}$ Checks

$c = g \star x_b$

Removing the TTP

Third try

Public parameters
$(G, X, \star)$

 generates x_0, x_1 in different orbits.

 b $\xleftarrow{x_0, x_1}$ 

Commit phase:

g
$c = g \star x_b$ $\xrightarrow{c}$ c

Opening phase: $\xrightarrow{\text{b}, g}$ Checks $c = g \star x_b$

Removing the TTP

Third try

Public parameters
$(G, X, \star)$

 generates x_0, x_1 in different orbits.

Issue

Interaction!

As in [BY91, JQSY19].

b

$$\xleftarrow{x_0, x_1}$$

Commit phase:

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$c = g \star x_b$

$$\xrightarrow{c}$$

c

Opening phase:

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Checks
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Given $(G, X, \star)$, we need a way to certify that two elements lie in different orbits.

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We call the tuple $(G, X, \star, f, \langle \cdot \rangle)$ *Group Action with Canonical Elements* (GACE).

Bit commitment from GACE

b

Public parameters

$(G, X, \star, f, \langle \cdot \rangle)$

and t_0, t_1 in T'.

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$\xrightarrow{\quad c \quad}$

c

Opening phase:

$\xrightarrow{\quad \text{b}, g \quad}$

Checks
$c = g \star \langle t_b \rangle$

Computationally hiding under the Pseudorandom assumption.
Perfectly binding.

A concrete instantiation (tensors!)

Let $X = \mathbb{F}_q^n \otimes \mathbb{F}_q^n \otimes \mathbb{F}_q^n$ and $G = \mathrm{GL}(n,q)^3$

$$(A,B,C) \star \sum_{ijk} S_{ijk} e_i \otimes e_j \otimes e_k = \sum_{ijk} S_{ijk} A e_i \otimes B e_j \otimes C e_k$$

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The invariant function is the tensor rank (NP-hard to compute)

$$f = \mathrm{rank}$$

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$(G, X, \star, \mathrm{rank}, \langle \cdot \rangle)$ is a Group Action with Canonical Elements.

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- We presented the first non-interactive bit commitment from cryptographic group actions.

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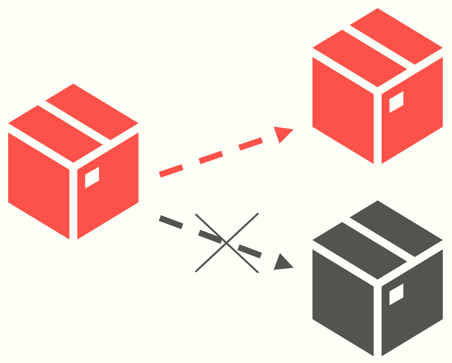

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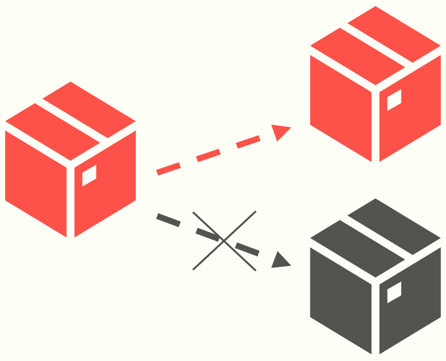

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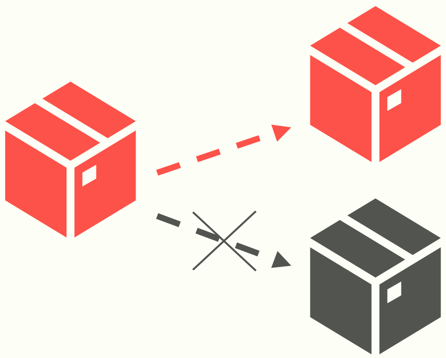

https://ia.cr/2023/723

Thanks!