On algebraically closed groups

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Algebraically closed groups

The theory of algebraically closed groups has an interesting history of ever growing connections between many different mathematical disciplines.

Moreover, algebraically closed groups are a great source of strange examples.

Among the others:

- Infinite Group Theory (B. H. Neumann, W. R. Scott);
- Model Theory (MacIntyre);
- Recursion Theory (Ziegler);
- Set Theory (Hickin);
- Game Theory (Ziegler);
- Homology (R.E. Phillips);
- And much more.

Equations over fields

Let F be a field and let c be an element of F. Does the equation

$$x^2 = c$$

have a solution in F?

Equations over fields

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Equations over fields

Problem

Given an equation $f(x) = 0$ over a field, find if the equation is soluble **over** F, namely if there is an element a in an algebraic extension F_1 of F such that $f(a) = 0$.

Equations over fields

Let F be a field and let c be an element of F. Does the equation

$$x^2 = c$$

have a solution in F?

Everyone knows, for example, that $x^2 = 2$ is soluble over $\mathsf{GF}(5)$ but not in $\mathsf{GF}(5)$.

Equations over groups

Let G be a group and let g be an element of G. Does the equation

$$x^2 = g$$

have a solution in G?

Equations and inequalities over groups

Let F be a free group on the countable set $\{x_1, x_2, \ldots\}$ and let G be a group. For any $w \in G * F$, the formula "$w = 1$" is said to be an *equation* over G and "$w \neq 1$" is said to be an *inequality* over G. The elements of $G * F$ are also called *terms* over G.

A set $\{w_i = 1, w_j \neq 1 \mid i \in I \text{ and } j \in J\}$ of equations and inequalities over G is *soluble over* G if there exists a group H containing G and a homomorphism

$$\varphi : G * F \longrightarrow H$$

such that

$$g^{\varphi} = g, \quad w_i \in \ker \varphi \quad \text{and} \quad w_j \notin \ker \varphi$$

for every $g \in G$, $i \in I$ and $j \in J$.

The set is said to be *soluble in* G if we can take $G = H$.

Equations and inequalities over groups

Some examples:

- The set $\{x^{-1}gx = g\}$ (here obviously w is $g^{-1}x^{-1}gx$) is soluble in G for any group G and any element $g \in G$.
- For any choice of the elements g, h and k of a group G, the set
$$\{x^{-1}gx = g, x^{-1}hx = h, x^{-1}kx \neq k\}$$
cannot be soluble over G if $k = gh$.
- The set above is soluble over G but not in G if $C_G(g) \cap C_G(h) \leqslant C_G(k)$ and $k \notin \langle g, h \rangle$.

  Take, for instance, $K = G * \langle x \rangle$, $N = \langle [g, x], [h, x] \rangle^K$ and $H = K/N$. Then G embeds into H and the above set has a solution in H.

Algebraically closed fields

$$\Downarrow$$

Algebraically closed groups

Algebraically closed groups

A group G is said to be *algebraically closed* if every finite set of equations defined over G that is soluble over G is soluble in G[1].

Let G be a group and let $\mathcal{P}$ be the propositional calculus generated by the propositions of the form "w=1", where w is a term over G. G is said to be *existentially closed* if every finite formula in $\mathcal{P}$ that is satisfiable in some group containing G is satisfiable in G.

[1] W.R. Scott: "Algebraically closed groups". *Proc. Am. Math. Soc.* 2, 118–121 (1951)

Algebraically and existentially closed groups

Theorem[2]

Let G be a group. Then the following are equivalent:

- G is existentially closed;
- Every finite set of equations and inequalities defined over G that is soluble over G is soluble in G;
- G is a non-trivial algebraically closed group.

[2]See G. Higman, E. Scott: "Existentially Closed Groups", LMS Monographs. New Series, vol. 3, *Oxford University Press*, New York, 1988.

Some properties

Existentially closed groups are the ”everything (reasonable) is possible” groups, so results about something not happening for them are usually more complex and deep.
Let’s look at some properties:

- Every e.c. group is simple and divisible;
- Every group G_0 can be embedded in an e.c. group G such that $|G| = \max(\aleph_0, |G_0|)$;
- There exist $2^{\aleph_0}$ pairwise non-isomorphic countable e.c. groups;
- No e.c. group is recursively presented;
- A finitely generated group can be embedded in every algebraically closed group if and only if it has soluble word problem.

The word problem for groups

Let $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finitely presented group. We say that (the presentation of) G has *soluble word problem* if there is an algorithm which, when a word w in the x_i is given, decides whether or not $w = 1$ is in G.

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Existentially closed groups are crazy

Theorem[3]

There exist $2^{\aleph_0}$ pairwise non-isomorphic countable e.c. groups $\{G_\alpha \mid \alpha \in 2^{\aleph_0}\}$ such that, for any $\alpha \neq \beta$, a finitely generated group is embeddable both into G_α and G_β if and only if it has soluble word problem.

This is somehow surprising, provided that two countable e.c. groups with the same finitely generated subgroups are always isomorphic.

[3]See G. Higman, E. Scott: "Existentially Closed Groups", LMS Monographs. New Series, vol. 3, *Oxford University Press*, New York, 1988.

To higher cardinalities

We now increase the cardinality and see what happens, to get new, interesting examples.

To higher cardinalities

Let κ be an infinite cardinal and let G be a group with $|G| \geqslant \kappa$. We say that G is κ-*existentially closed* if every system of less than κ-many equations and inequalities defined over G that is soluble over G is soluble in G[4].

Obviously, if $\kappa = \aleph_0$ we get again the classical existentially closed groups.

[4] W. R. Scott: "Algebraically closed groups". *Proc. Am. Math. Soc.* 2, 118–121 (1951)

On κ-e.c. groups

Question

In which ways κ-e.c. groups differ from e.c. groups?

Some properties for κ-e.c. groups

Theorem[5]

Let G be a group and let κ be an uncountable cardinal. Then G is κ-existentially closed if and only if

(a) G contains an isomorphic copy of every group of cardinality less than κ,

(b) every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.

[5]Otto H. Kegel, Mahmut Kuzucuoğlu , *κ-existentially closed groups*, J. Algebra 499, 298-310, (2018).

Recall

Theorem

There exist $2^{\aleph_0}$ pairwise non-isomorphic countable e.c. groups.

Some properties for κ-e.c. groups

Theorem[6]

If $\kappa > \aleph_0$, any two κ-existentially closed groups of cardinality κ are isomorphic.

[6] O. H. Kegel, M. Kuzucuoğlu: *κ-existentially closed groups*, J. Algebra 499, 298-310, (2018).

Recall

Theorem

Every infinite group can be embedded in an e.c. group of the same cardinality. In particular, there are e.c. groups of cardinality κ for any infinite cardinal κ.

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Some properties for κ-e.c. groups

Theorem[7]

If $\kappa > \aleph_0$ is a singular cardinal, then there exists no κ-existentially closed group of cardinality κ.

[7] B. Kaya, O. H. Kegel and M. Kuzucuoğlu: *On the existence of κ-existentially closed groups*, Arch. Math. (Basel), 111, 225-229, (2018).

On κ-e.c. groups of higher cardinality

Theorem

Let κ be a regular cardinal for which there exists a κ-e.c. group G. Then, we find 2^{κ^+} non-isomorphic κ-e.c. groups of cardinality κ^+.

Obviously the number of isomorphism classes is the maximum possible. In this way, we recover Neumann results, which does not hold for κ, for κ^+.

Existentially closed groups are crazy

Theorem

Let G be an e.c. group. Then we may find in G a family of subgroups $\{X_n\}$ (all with the same propositional calculus of G), which, ordered by inclusion, has the same order type as the rational numbers. In particular, if $|G| = \aleph_0$, the same chain can be made from subgroups isomorphic with G.

Also κ-existentially closed groups are crazy

Theorem

Let G be a κ-e.c. group. Then we may find in G a family of subgroups $\{X_n\}$ (all with the same propositional calculus of G), which, ordered by inclusion, has the same order type as the rational numbers. In particular, if $|G| = \kappa$, the same chain can be made from subgroups isomorphic with G.

On κ-e.c. groups of higher cardinality

Question

What kind of ordered families of subgroups of κ-e.c. groups can we find?

[Hint: It has to deal with the so-called κ-density and, without the Continuum Hypothesis, is related to the Baumgartner's axiom.]