GRAPH-RESTRICTIVE AND EXPONENT-RESTRICTIVE ACTIONS

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Young Researcher Algebra Conference 2023

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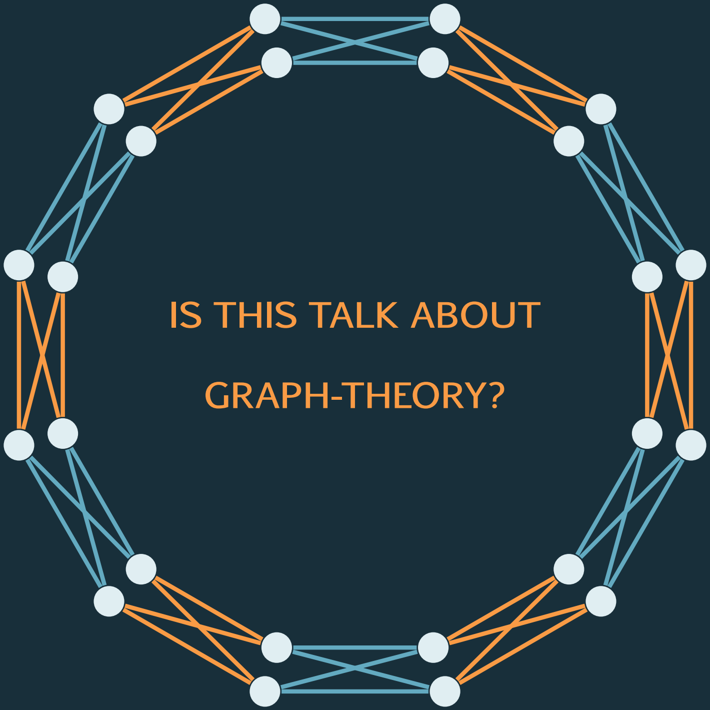
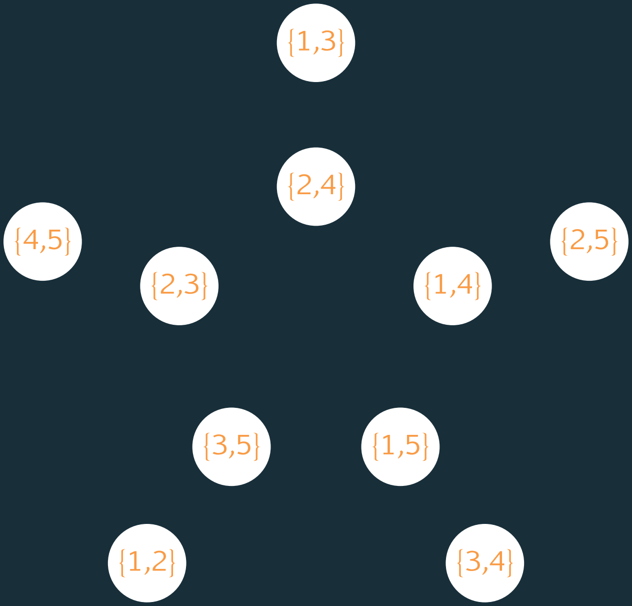
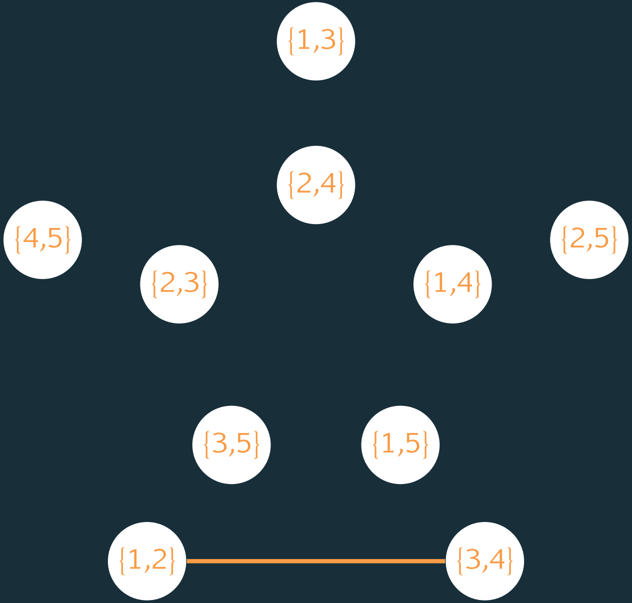


IS THIS TALK ABOUT GRAPH-THEORY?



Suppose that $\mathrm{Alt}(5)$ is acting on the 2-subsets of $\{1, 2, 3, 4, 5\}$.



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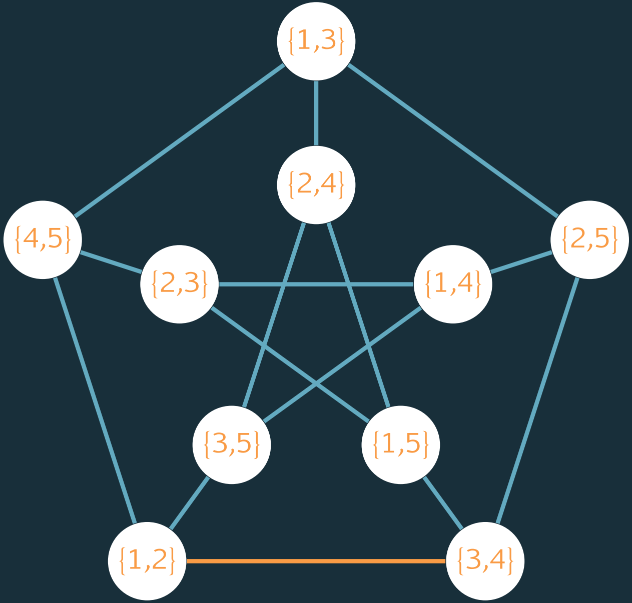

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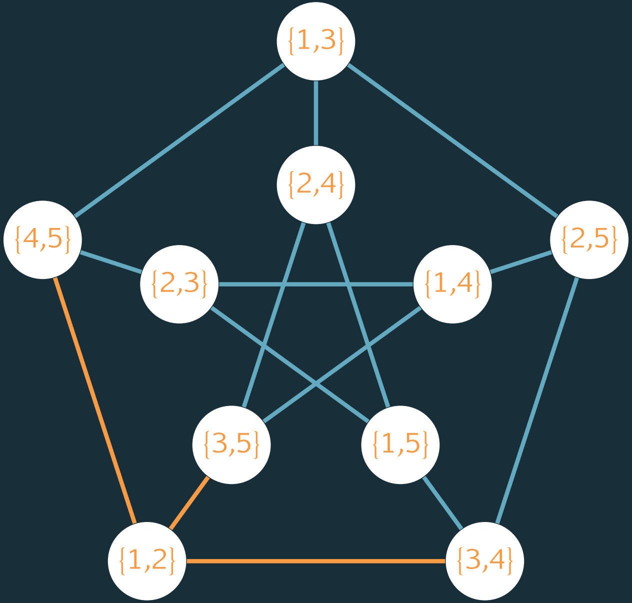

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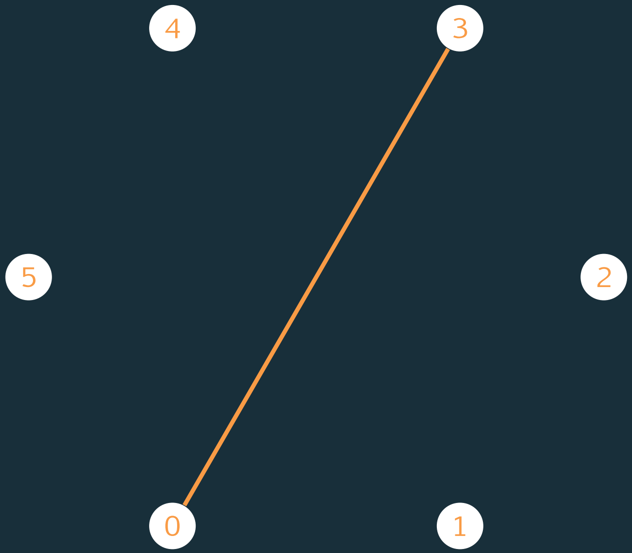
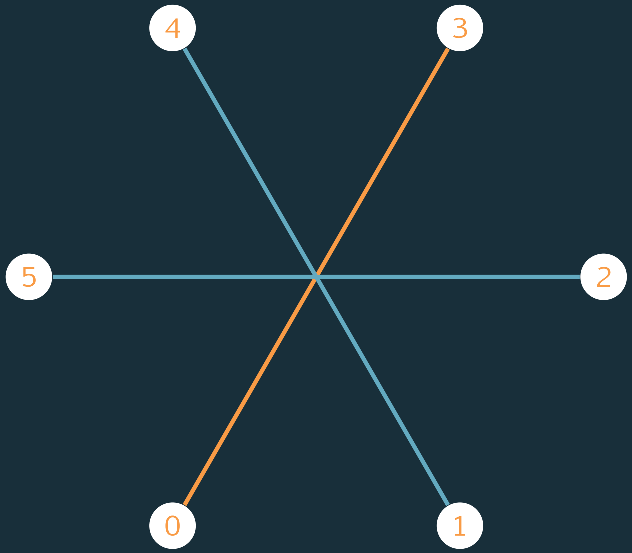


Suppose that C_6 is acting regularly on 6 points.



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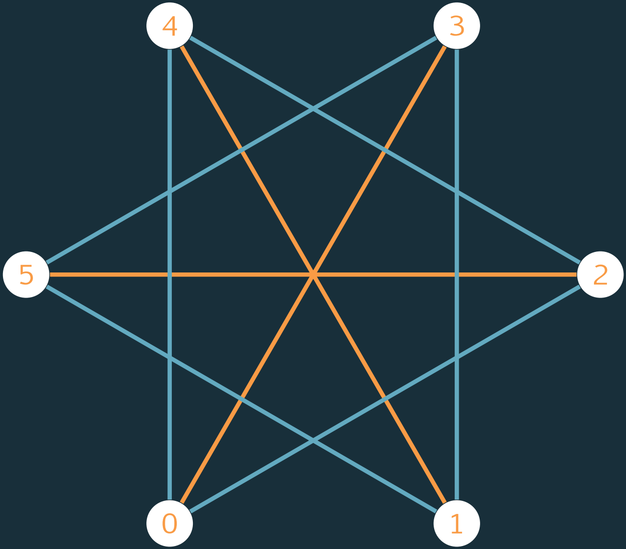
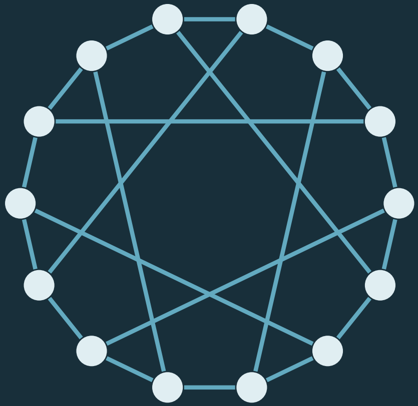


Suppose that C_6 is acting regularly on 6 points.

A pair (Γ, G) is a **vertex-transitive graph** if

- Γ is a finite simple connected graph,
- the action of G on the vertices of Γ is faithful and transitive,
- the action of G preserves the adjacency relation.



The **local action of (Γ, G)** is the action that the **vertex-stabilizer** G_α induces on the **neighbourhood** $\Gamma(\alpha)$, that is,

$$G_\alpha^{\Gamma(\alpha)} = \frac{G_\alpha}{G_{(\alpha \cup \Gamma(\alpha))}}.$$

AND NOW FOR SOMETHING

COMPLETELY DIFFERENT

Let L be a permutation group. We say that the pair (Γ, G) is **locally-L** if the local action is isomorphic to L, that is,

$$G_\alpha^{\Gamma(\alpha)} = L.$$

We say that L is **graph-restrictive** if there exists a constant $\mathbf{c}(L)$ such that,

$$\text{for any locally-}L\text{ pair }(\Gamma, G), \quad |G_\alpha| \leq \mathbf{c}(L).$$

- ♣ G. Verret, On the order of arc-stabilizers in arc-transitive graphs. *Bulletin of the Australian Mathematical Society* **80** (2009) pp. 498–505.
- ♣ P. Potočnik, P. Spiga, G. Verret, On graph-restrictive permutation groups. *Journal of Combinatorial Theory, Series B* **112** (2012) pp. 820–831.

Theorem (W. T. Tutte – 1959)

Let (Γ, G) be a **3-valent** vertex-transitive graph. Suppose that the local action L is transitive. Then

$$|G_\alpha| \leq 48.$$

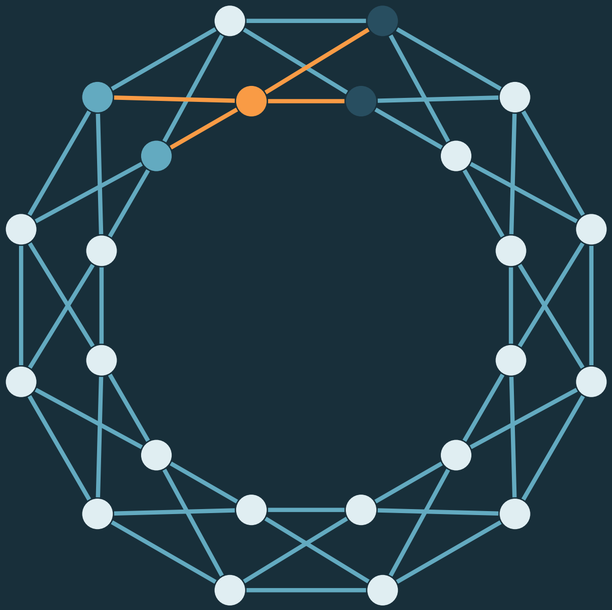
♣ W. T. Tutte, On the symmetry of cubic graphs. *Canadian Journal of Mathematics* **11** (1959) pp. 621–624.

In particular, C_3 and $\mathrm{Sym}(3)$ in their primitive actions of degree 3 are graph-restrictive.

Conjecture (R. Weiss – 1978)

Any **primitive** group L is graph-restrictive.

♣ R. Weiss, s-transitive graphs. *Colloquia Mathematica Societatis János Bolyai* **25** (1978) pp. 827–847.



Theorem (C. E. Praeger, L. Pyber, P. Spiga, E. Szabó – 2012)

Suppose that G has all its alternating section having degree at most r, and that L is primitive. Then, there exist a function f such that

$$\mathbf{c}(L) \leq f(r).$$

♣ C. E. Praeger, L. Pyber, P. Spiga, E. Szabó, Graphs with automorphism groups admitting composition factors of bounded rank. *Proceedings of the American Mathematical Society* **140** (2012) pp. 2307–2318.

Suppose that (Γ, G) is a locally-L pair, and that L is graph-restrictive. Then

- the order of G is bounded from above by a linear function of the number of vertices, that is,

$$|G| \leq |G_\alpha||V\Gamma| \leq \mathbf{c}(L)|V\Gamma|;$$

- the minimal number of generators for G is bounded from above by a constant, that is,

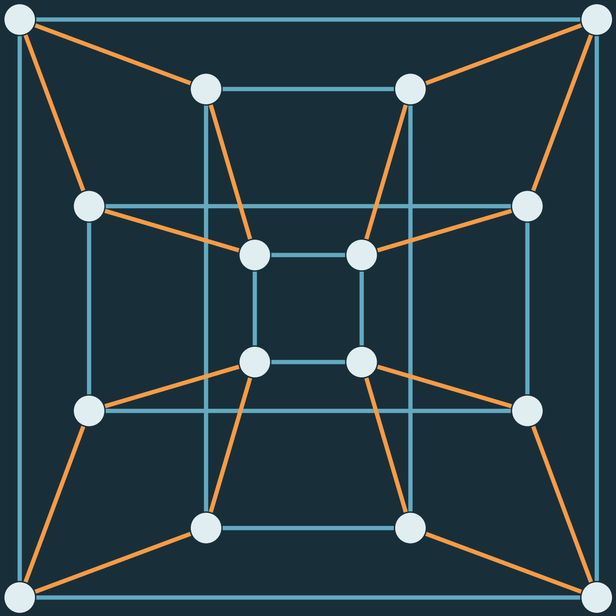
$$\mathbf{d}(G) \leq |\Gamma(\alpha)| + |G_\alpha| \leq |\Gamma(\alpha)| + \mathbf{c}(L);$$

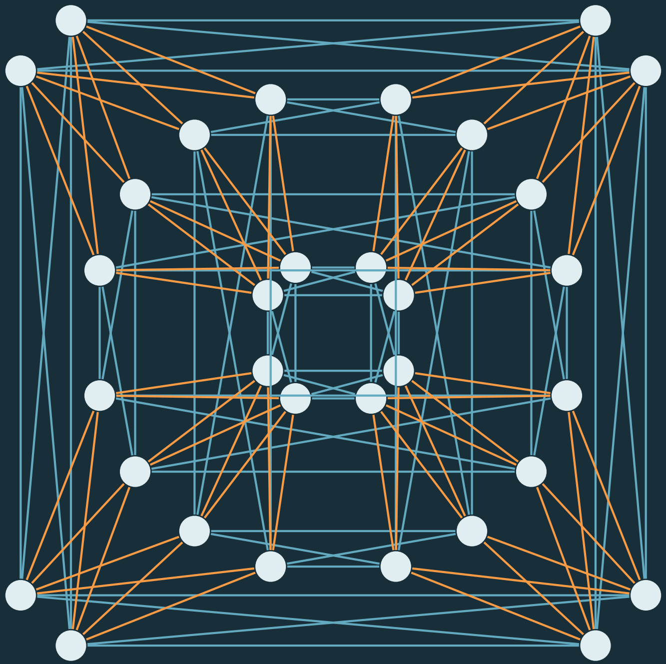
- the exponent of G_α is bounded from above by a constant, that is,

$$\exp(G_\alpha) \leq |G_\alpha| \leq \mathbf{c}(L).$$

Theorem (M. B., V. Grazian, P. Spiga)

There are infinite families of pairs (Γ_h, G_h), whose common local action L is transitive and imprimitive, such that $\mathbf{d}(G_h)$ **cannot be bounded from above by a constant**.





Let L be a permutation group. We say that L is **exponent-restrictive** if there exists a constant $\mathbf{e}(L)$ such that,

$$\text{for any locally-}L\text{ pair } (\Gamma, G), \quad \exp(G_\alpha) \leq \mathbf{e}(L).$$

Question (P. Spiga – 2012)

Is any group L exponent-restrictive?

All the explicit examples we know are such that

$$\exp(G_\alpha) \leq \exp(L)^2.$$

The known families of exponent-restrictive groups are

- the graph-restrictive groups,
- the groups of degree 3,
- the transitive groups of degree at most 5.

Theorem (M. B., P. Potočnik, P. Spiga)

The dihedral groups (in their natural action) are exponent-restrictive.

THANK YOU FOR

YOUR ATTENTION!