Some recent results on the conformal superalgebra CK_6

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Definition (Conformal superalgebra)

A **conformal superalgebra** R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with a $\mathbb{C}-$linear map, called $\lambda-$bracket, $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto [a_\lambda b]$, that satisfies the following properties for all $a, b, c \in R$:

1. $[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b];$
2. $[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial} a];$
3. $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]];$

where $p(a)$ denotes the parity of the element $a \in R$ and $p(\partial a) = p(a)$ for all $a \in R$.

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A module M over a conformal superalgebra R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with the $\mathbb{C}-$linear map $R \otimes M \to \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a_\lambda v$ that satisfies the following properties for all $a, b \in R$, $v \in M$:

1. $(\partial a)_\lambda v = [\partial, a_\lambda]v = -\lambda a_\lambda v$;
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A module M is called **finite** if it is a finitely generated $\mathbb{C}[\partial]-$module.

Remark

We recall that it is possible to associate with a conformal superalgebra R a Lie superalgebra $\mathcal{A}(R)$ that plays a fundamental role in the representation theory of R.

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Theorem (Fattori, Kac 2002)

Any finite simple conformal superalgebra R is isomorphic to one of the conformal superalgebras of the following list: $\mathrm{Cur}\,\mathfrak{g}$, *where $\mathfrak{g}$ is a simple finite−dimensional Lie superalgebra,* $W_n (n \geq 0)$, $S_{n,b}$, $\tilde{S}_n$ $(n \geq 2,\ b \in \mathbb{C})$, $K_n (n \geq 0,\ n \neq 4)$, K'_4, CK_6.

The conformal superalgebra of type K

Let $\bigwedge(N)$ be the Grassmann superalgebra in the N odd indeterminates $\xi_1, ..., \xi_N$.
Let t be an even indeterminate and $\bigwedge(1,N) = \mathbb{C}[t, t^{-1}] \otimes \bigwedge(N)$.

$$W(1,N) = \left\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1,N) \right\}.$$

Let $\omega = dt - \sum_{i=1}^{N} \xi_i d\xi_i$.

$$K(1,N) = \{ D \in W(1,N) \mid D\omega = f_D \omega \text{ for some } f_D \in \bigwedge(1,N) \}.$$

We can define $\bigwedge(1,N)_+ = \mathbb{C}[t] \otimes \bigwedge(N)$, $W(1,N)_+$ and $K(1,N)_+$.

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$\Lambda(1,N)$ has a Lie superalgebra structure as follows: for all $f, g \in \Lambda(1,N)$

$$[f,g] = \Big(2f - \sum_{i=1}^{N} \xi_i \partial_i f\Big)(\partial_t g) - (\partial_t f)\Big(2g - \sum_{i=1}^{N} \xi_i \partial_i g\Big) + (-1)^{p(f)}\Big(\sum_{i=1}^{N} \partial_i f \partial_i g\Big).$$

$K(1,N) \cong \Lambda(1,N)$ as Lie superalgebras via:

$$\Lambda(1,N) \longrightarrow K(1,N)$$

$$f \longmapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_t f + \partial_i f)(\xi_i \partial_t + \partial_i).$$

We consider on $K(1,N)$ the standard grading, i.e. $\deg(t^m \xi_{i_1} \cdots \xi_{i_s}) = 2m + s - 2$.

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The conformal superalgebra of type K is defined as

$$K_N := \mathbb{C}[\partial] \otimes \bigwedge(N).$$

For $f = \xi_{i_1} \cdots \xi_{i_r}$ and $g = \xi_{j_1} \cdots \xi_{j_s}$:

$$[f_\lambda g] = \big((r-2)\partial(fg) + (-1)^r \sum_{i=1}^{N} (\partial_i f)(\partial_i g)\big) + \lambda(r+s-4)fg.$$

We recall that

$$\mathfrak{g} = \mathcal{A}(K_N) = K(1,N)_+.$$

$\mathfrak{g}$ has depth 2 with respect to the standard grading. $\mathfrak{g}_{-2}$ is one$-$dimensional, we call Θ the generator $-1/2$ of $\mathfrak{g}_{-2}$.

Let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}-$graded Lie superalgebra. We will use the notation $\mathfrak{g}_+ = \oplus_{i>0} \mathfrak{g}_i$, $\mathfrak{g}_- = \oplus_{i<0} \mathfrak{g}_i$ and $\mathfrak{g}_{\geq 0} = \oplus_{i \geq 0} \mathfrak{g}_i$.

Remark

Let F be a $\mathfrak{g}_{\geq 0}-$module. We denote by $M(F)$ the **generalized Verma module**. If F is a finite$-$dimensional irreducible $\mathfrak{g}_{\geq 0}-$module, we call $M(F)$ a **finite Verma module**.
If $M(F)$ is not irreducible, we call $M(F)$ **degenerate**.
We have a $\mathbb{Z}_{\geq 0}-$grading on $U(\mathfrak{g}_-)$ and $M(F)$.

Definition

Given a $\mathfrak{g}-$module V, we call **singular vectors** the elements of:

$$\mathrm{Sing}(V) = \{ v \in V \mid \mathfrak{g}_+ . v = 0 \} .$$

If $V = M(F)$, we will call **trivial singular vectors** the singular vectors of degree 0 and **nontrivial singular vectors** the singular vectors of positive degree.

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By a series of results obtained by Boyallian, Kac, Liberati (2010), Kac, Rudakov (2002) and Cheng, Lam (2001), it is known that for a conformal superalgebra of type K the classification of finite irreducible modules can be obtained through the classification of all degenerate finite Verma modules for the associated annihilation superalgebra; moreover, this is equivalent to the classification of all (highest weight) singular vectors in finite Verma modules.

The conformal superalgebra CK_6

For $\xi_I \in \Lambda(6)$ we define ξ_I^* to be such that $\xi_I \xi_I^* = \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6$.

$$CK_6 = \mathbb{C}[\partial] - \text{span}\left\{ f - i(-1)^{\frac{|f|(|f|+1)}{2}}(-\partial)^{3-|f|} f^*,\ f \in \Lambda(6), 0 \leq |f| \leq 3 \right\}.$$

Remark

We recall that

$$\mathfrak{g} := \mathcal{A}(CK_6) \cong E(1,6).$$

The homogeneous components of non−positive degree of $\mathfrak{g}$ and $K(1,6)_+$ coincide and are:

$$\mathfrak{g}_{-2} = \langle 1 \rangle,$$
$$\mathfrak{g}_{-1} = \langle \xi_1, \xi_2, ..., \xi_6 \rangle,$$
$$\mathfrak{g}_0 = \langle t, \xi_i\xi_j :\ 1 \leq i,j \leq 6 \rangle \cong \mathbb{C}t \oplus \mathfrak{sl}(4).$$

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Remark

Let F be an irreducible finite-dimensional $\mathfrak{g}_0-$module. Then

$$M(F) \cong \mathbb{C}[\Theta] \otimes \wedge(6) \otimes F.$$

Lemma (B.; statement by Boyallian, Kac, Liberati 2013)

Let $\vec{m} \in M(F)$ is a singular vector. Then the degree of $\vec{m}$ with respect to Θ is at most 2.

From now on we denote the highest weight of an irreducible finite−dimensional $\mathfrak{g}_0-$module as $\mu = (n_1, n_2, n_3, \mu_t)$ where μ_t is the weight with respect to h_1, h_2, h_3 and t, where

$$h_1 = -i\xi_{34} - i\xi_{56}, \quad h_2 = -i\xi_{12} + i\xi_{34}, \quad h_3 = -i\xi_{34} + i\xi_{56}.$$

Boyallian, Kac and Liberati classified all highest weight singular vectors for CK_6 and obtained the following morphisms between degenerate finite Verma modules.

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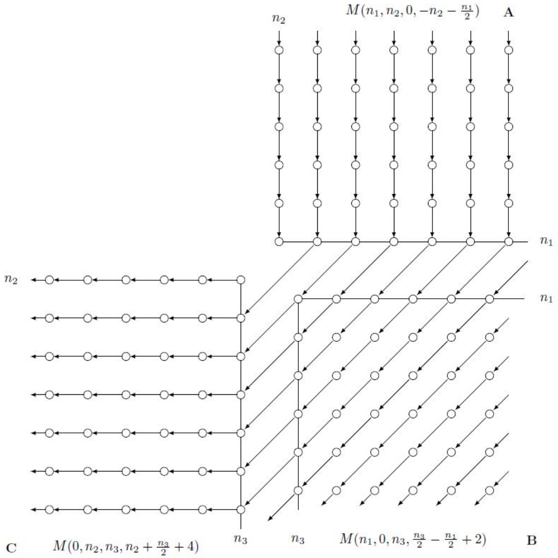
Remark

Between $M(\mu)$ and $M(\tilde{\mu})$ there exists a morphism of $\mathfrak{g}-$modules if and only if there exists a non trivial singular vector $\vec{m}$ in $M(\tilde{\mu})$ of highest weight μ.

$$\nabla : M(\mu) \longrightarrow M(\tilde{\mu})$$
$$v_\mu \longmapsto \vec{m}$$

If $\vec{m}$ is a singular vector of degree d, we say that ∇ is a morphism of degree d.

Boyallian, Kac, Liberati 2013:



Proposition (B.)

As $\mathfrak{g}_0-$modules:

$$H^{n_1,n_2}(M_A) \cong \begin{cases} \mathbb{C} & \textit{if } (n_1, n_2) = (0,0), \\ 0 & \textit{otherwise.} \end{cases}$$

$$H^{n_2,n_3}(M_C) \cong \begin{cases} \mathbb{C} & \textit{if } \ (n_2, n_3) = (1,0), \\ 0 & \textit{otherwise.} \end{cases}$$

Open problems

- It remains to complete the study of the homology of the complexes for the second quadrant of CK_6;
- it would be interesting to understand if it is possible to use similar techniques to compute the homology of the complexes for $E(5, 10)$.