Università degli Studi della Basilicata
Dipartimento di Matematica, Informatica ed Economia

Lie semisimple algebras of derivations and varieties of PI algebras with almost polynomial growth

Sebastiano Argenti
sebastiano.argenti@unibas.it

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Polynomial Identities

Let $F\langle X\rangle$ be the free algebra over F on the countable set of non commuting variables $X = \{x_1, x_2, \dots\}$.

Definition

Let A be an F-algebra. A polynomial $f = f(x_1, \dots, x_n) \in F\langle X\rangle$ is a polynomial identity for A if, for each $a_1, \dots, a_n \in A$,

$$f(a_1, \dots, a_n) = 0$$

Example

Commutative algebras satisfy the commutator identity

$$[x_1, x_2] = x_1x_2 - x_2x_1$$

Codimension sequence

A polynomial $f \in F\langle X \rangle$ is multilinear if it is linear in each variable.

$$P_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)}$$

$$c_n(A) = \dim P_n(A)$$

Theorems on the codimensions growth

Let A be a PI algebra, then

- Regev, 1972 [1]: $c_n(A)$ is exponentially bounded.
- Kemer, 1991 [2]: $c_n(A)$ either grows exponentially or is polynomially bounded.
- Giambruno, Zaicev, 1999 [3]: $\exists \exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{N}$

Almost polynomial growth

The variety $\mathcal{V} = \text{var}(A)$ generated by A is the class of all the algebras satisfying all the identities of A.

Definition

We say that the variety $\mathcal{V}$ has almost polynomial growth if $c_n(\mathcal{V})$ has exponential growth and for each proper subvariety $\mathcal{U} < \mathcal{V}$, $c_n(\mathcal{U})$ is polynomially bounded.

Theorem (Kemer [2])

The only varieties with almost polynomial growth are those generated by the algebra $UT_2(F)$ of upper triangular matrices or by the Grassmann algebra E.

Other classifications

The theory of PI algebras has been extended to algebras with involution, superalgebras, graded algebras, trace algebras, differential algebras, etc..

Theorem

The varieties with almost polynomial growth are only those generated by...

- Algebras with involution (Giambruno, Mischenko [4]):
  $F \oplus F$, M
- Superalgebras (Giambruno, Mischenko, Zaicev [5]):
  $F \oplus tF$, UT_2, UT_2^{gr}, E, E^{gr}
- Graded algebras (Valenti [6]):
  $F[C_p]$, UT_2^G, E, $E^{\mathbb{Z}_2}$
- Finite dimensional trace algebras (Ioppolo, Koshlukov, La Mattina [7]):
  UT_2, $D_2^{t_{\alpha,\beta}}$,$D_2^{t_{\gamma,\gamma}}$, $D_2^{t_{\delta,0}}$, $C_2^{t_{\epsilon,1}}$
- Differential algebras with solvable Lie algebra action (Rizzo [8]):
  UT_2, UT_2^{ε}

Differential algebras

A derivation on the algebra A is a linear map $\delta : A \to A$ such that for each $a, b \in A$

$$(ab)^\delta = a^\delta b + ab^\delta$$

The set $Der(A)$ of all the derivations of A is a Lie algebra.

An action of the Lie algebra L on A by derivations is a Lie homomorphism $\rho : L \to A$. For each $\delta \in L, a \in A$ we write

$$\rho(\delta) : a \longmapsto a^\delta$$

We say that A is an algebra with derivations or an L-differential algebra.

Every L-action on A induces an algebra homomorphism from the universal enveloping algebra $\mathcal{U}(L)$ and $End(A)$.

Differential polynomials

Let $X = \{x_1, \ldots, x_n\}$ be a countable set of variables and denote with V_X the free vector space spanned by X.
The free L-differential algebra is

$$F_L\langle X\rangle = \bigoplus_{n=0}^{\infty}(V_X \otimes \mathcal{U}(L))^{\otimes n}$$

The elements of $F_L\langle X\rangle$ can be thought as polynomials in the variables $x^u = x_i \otimes u \in V_X \otimes \mathcal{U}(L)$, $x \in X, u \in \mathcal{U}(L)$.

L has a natural action on $F_L\langle X\rangle$ such that $(x^u)^\delta = x^{u\delta}$ for each $x \in X, u \in \mathcal{U}(L), \delta \in L$.

A differential identity for A is a polynomial in $F_L\langle X\rangle$ which vanishes under every substitution of the variables with elements of A.

Differential identities

$$P_n^L = \mathrm{span}_F\{x_{\sigma(1)}^{d_1} \cdots x_{\sigma(n)}^{d_n} \mid \sigma \in S_n, d_1, \ldots, d_n \in \mathcal{U}(L)\}$$

If A is finite dimensional then $P_n^L(A)$ is finite dimensional too. Therefore is well defined $c_n^L(A) = \dim P_n^L(A)$.

Theorem (Gordienko, Kochetov [9])

Let A be a finite dimensional non-nilpotent L-differential algebra over a field F of characteristic 0. Suppose L is finite dimensional semisimple. Then there exist constants $C_1, C_2 > 0, r_1, r_2 \in \mathbb{R}, d \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$C_1 n^{r_1} d^n \le c_n^L(A) \le C_2 n^{r_2} d^n$$

Thus the differential codimension sequence of a finite dimensional L-algebra is either polynomially bounded or grows exponentially.

Two special classes of algebras

Let W be a representation of L, or L-module, i.e. a vector space equipped with a Lie homomorphism $\rho : L \to End(W)$.

$UT_2(W)$

The triangular algebra $\begin{pmatrix} F & W \\ 0 & F \end{pmatrix}$ with L action given by

$$\begin{pmatrix} \alpha & v \\ 0 & \beta \end{pmatrix}^{\delta} = \begin{pmatrix} 0 & v^{\delta} \\ 0 & 0 \end{pmatrix} \qquad \forall \alpha, \beta \in F, v \in W, \delta \in L$$

$End(W)$

The algebra of all the the W-endomorphisms with L action given by

$$f^{\delta} = [f, \rho(\delta)] = f \circ \rho(\delta) - \rho(\delta) \circ f \qquad \forall f \in End(W), \delta \in L$$

Irreducible representations

Weyl's Theorem

Let L be a semisimple Lie algebra over a field of characteristic zero. Then every finite dimensional representation of L decomposes as a direct sum of irreducible subrepresentations.

Theorem (Classification of the irreducible representations)

Let L be a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero and let $\Lambda \subseteq \mathfrak{h}^*$ be the set of integral dominant weights. Then

- for each $\lambda \in \Lambda$ there exists a finite dimensional irreducible L-module W_λ which has highest weight λ.
- $W_\lambda \approxeq W_\mu$ as L-modules if and only if $\lambda = \mu$, $\lambda, \mu \in \Lambda$.
- every finite dimensional irreducible L-module is isomorphic to a W_λ for some $\lambda \in \Lambda$.

Main Theorem

Theorem

Let L be a semisimple Lie algebra with finite dimensional irreducible modules W_λ, $\lambda \in \Lambda$. Let A be a finite dimensional algebra on which L acts as derivations.
Then $\mathcal{V} = \text{var}^L(A)$ has exponential growth if and only if it contains one the algebras $UT_2(W_\lambda)$ or $End(W_\mu)$ with $\lambda, \mu \in \Lambda, \mu \neq 0$.

Corollary

Let $\mathcal{V} = \text{var}^L(A)$ be a variety of differential algebras generated by a finite dimensional algebra. If $\mathcal{V}$ has almost polynomial growth then it is generated by one of the algebras $UT_2(W_\lambda)$ or $End(W_\mu)$ with $\lambda, \mu \in \Lambda, \mu \neq 0$.

The Lie algebra $L = \mathfrak{sl}_2$

$$\mathfrak{sl}_2 = \text{span}_F\{h, e, f\}$$
$$[e, h] = 2e \qquad [f, h] = -2f \qquad [f, e] = h$$

The set of integral dominant weights of $\mathfrak{sl}_2$ is $\Lambda = \mathbb{N} = \{0, 1, 2, \dots\}$.
Accordingly $\mathfrak{sl}_2$ has irreducible modules $V_0, V_1, \dots$
Moreover $\dim V_n = n + 1$ for each $n \in \mathbb{N}$.

The Casimir element Ω

$$\Omega = \frac{1}{2}h^2 + ef + fe \in \mathcal{U}(\mathfrak{sl}_2)$$

Ω acts on the irreducible representations V_n by multiplication with the scalar

$$\lambda_n = \frac{n(n+2)}{2} \qquad n = 0, 1, \dots$$

Classification problem for $\mathfrak{sl}_2$

Theorem

If $\mathcal{V} = \text{var}^{\mathfrak{sl}_2}(A)$, $\dim A < \infty$, has almost polynomial growth then it is generated by one of the algebras $UT_2(V_n)$ or $End(V_m)$ with $n \geq 0$, $m \geq 1$.

Conjecture

The $\mathfrak{sl}_2$-differential varieties with almost polynomial growth are only those generated by $UT_2(V_n)$ or $End(V_m)$ with $n \geq 0$, $m \geq 1$.

Remark

The conjecture is true if we prove that for each
$A, B \in \{UT_2(V_n) | n \geq 0\} \cup \{End(V_m) | m \geq 1\}$ we have $Id_{\mathfrak{sl}_2}(A) \not\subseteq Id_{\mathfrak{sl}_2}(B)$.

$UT_2(V_n)$

Theorem

For each $n \in \mathbb{N}$ the algebra $UT_2(V_n)$ satisfies the differential identities

$$x_1^{\delta_1} x_2^{\delta_2}, \qquad \delta_1, \delta_2 \in \mathfrak{sl}_2$$
$$[x,y]^{\Omega} - \frac{n(n+2)}{2}[x,y]$$

where $\Omega = \frac{1}{2}h^2 + ef + fe \in \mathcal{U}(\mathfrak{sl}_2)$ is the Casimir operator.

Corollary

- For each $m, n \geq 0$, $UT_2(V_m) \notin \mathrm{var}^{\mathfrak{sl}_2}(UT_2(V_n))$.
- For each $m \geq 1, n \leq 0$, $End(V_m) \notin \mathrm{var}^{\mathfrak{sl}_2}(UT_2(V_n))$.

Therefore for each $n \in \mathbb{N}$ the algebra $UT_2(V_n)$ has almost polynomial growth.

Projections

Proposition

For each $N, i \in \mathbb{N}$ define

$$\pi_i^{(N)} = \prod_{\substack{j=0 \\ j\neq i}}^{N} \frac{\lambda_j - \Omega}{\lambda_j - \lambda_i}$$

Then for each $n \leq N$, the element $\pi_i^{(N)} \in \mathcal{U}(\mathfrak{sl}_2)$ acts on V_n as the zero map if $n \neq i$ and as the identity if $n = i$.

Remark

If W has highest weight N, then $\pi_i^{(N)}, \pi_i^{(N+1)}, \pi_i^{(N+2)} \dots$ have the same action on W so we can drop the superscript and simply write π_i.
It is the projection on the irreducible components isomorphic to V_i.

$End(V_n)$

As an $\mathfrak{sl}_2$-module we have the decomposition

$$End(V_n) \approxeq M_{n+1}(F) \approxeq F \oplus \mathfrak{sl}_{n+1} = V_0 \oplus V_2 \oplus \cdots \oplus V_{2n-2} \oplus V_{2n}$$

Theorem

For each $n \in \mathbb{N}$, the algebra $End(V_n)$ satisfy the differential identities

$$[x^{\pi_0}, y]$$
$$[x, y]^{\pi_0}$$

Corollary

For each $m \geq 0, n \geq 1,\ UT_2(V_m) \notin \mathrm{var}^{\mathfrak{sl}_2}(End(V_n))$.

Traces

Definition

A trace on an algebra A is defined as a linear map $\mathrm{tr} : A \to A$ such that

- $\mathrm{tr}(a)b = b\,\mathrm{tr}(a) \quad \forall a, b \in A$
- $\mathrm{tr}(ab) = \mathrm{tr}(ba) \quad \forall a, b \in A$
- $\mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\,\mathrm{tr}(b) \quad \forall a, b \in A$

Remark

The map π_0 is a trace on $End(V_n)$.
With the identification $End(V_n) \approxeq M_{n+1}(F)$, the usual trace corresponds to $(n+1)\pi_0$.
Therefore all the trace identities of the matrix algebra, such as the Cayley-Hamilton identity, are contained in the differential identities of $End(V_n)$.

Trace formulas

Theorem

Let $\rho : \mathfrak{sl}_2 \to End(V_n)$ be an irreducible representation. Then for each $\delta \in \mathfrak{sl}_2$, with $\delta = ah + be + cf$, $a, b, c \in F$, and for each $k = 0, 1, \ldots$ we have

$$\mathrm{tr}(\rho(\delta)^{2k+1}) = 0$$
$$\mathrm{tr}(\rho(\delta)^{2k}) = \alpha_{k,n}(a^2 + bc)^k$$

where the coefficients $\alpha_{k,n} \in F$ satisfy the recurrence relation

$$\alpha_{2k,n} = \frac{n^{2k+1} + (n+2)^{2k+1} - 2^{2k+1}(n+1)}{4k+2} - \sum_{j=1}^{k-1} \binom{2k}{2j-1} \frac{4^{k-j}}{2j} \alpha_{2j,n}$$

Trace formulas

The first few trace formulas are

$$\mathrm{tr}(\rho(\delta)^0) = n + 1$$

$$\mathrm{tr}(\rho(\delta)^2) = \frac{n(n+1)(n+2)}{3}(a^2 + bc)$$

$$\mathrm{tr}(\rho(\delta)^4) = \frac{n(n+1)(n+2)(3n^2+6n-4)}{15}(a^2 + bc)^2$$

Since $Im(\rho) = Im(\pi_2)$, we obtain the differential identity for $End(V_n)$

$$(x^{\pi_2}x^{\pi_2}x^{\pi_2}x^{\pi_2})^{\pi_1} - \frac{3}{5} \cdot \frac{(3n^2+6n-4)}{n(n+2)}(x^{\pi_2}x^{\pi_2})^{\pi_1}(x^{\pi_2}x^{\pi_2})^{\pi_1}$$

Classification

Corollary

For each $m, n \geq 1$, $End(V_m) \notin \mathrm{var}^{\mathfrak{sl}_2}(End(V_n))$.
Therefore for each $n \geq 1$ the algebra $End(V_n)$ has almost polynomial growth.

Theorem

The $\mathfrak{sl}_2$-differential varieties with almost polynomial growth are only those generated by $UT_2(V_n)$ or $End(V_m)$ with $n \geq 0$, $m \geq 1$.

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THANK YOU!