An "almost concise" overview of concise words

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Let $w = w(x_1, \ldots, x_n)$ be a group-word in variables $x_1, \ldots, x_n$,
G_w the (normal) set of all values of w in a group G,
$w(G)$ the verbal subgroup of the group G generated by G_w.

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- *Multilinear commutators* (*outer commutator words*):
  They are obtained by nesting commutators but using always different variables:

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The *lower central words* γ_k:

$$\gamma_1 = x_1, \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k], \quad \text{for } k \geq 2.$$

The corresponding verbal subgroups are the $\gamma_k(G)$.

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The *derived words* δ_k:

$$\delta_0 = x_1,\ \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})],\ k \geq 1.$$

The verbal subgroups are the $G^{(k)}$.

Examples:

- The *Engel words* $[x,_n y]$:

$$[x,_0 y] = x, \quad [x,_n y] = [[x,_{n-1} y], y], \quad \text{for } n \geq 1.$$

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- *Non-commutator words*: Words such that the sum of the exponents of some variable involved in it is non-zero.

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Question

May we get any information on $w(G)$ imposing some condition on G_w?

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R. F. Turner-Smith (1964)

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The n-th Engel word $[x, {}_n y]$ is concise for $n \leq 4$.

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The n-th Engel word $[x, {}_n y]$ is concise for $n \leq 4$.

But, conciseness of these words for $n \geq 5$ remains unknown!

Is every word concise?

Negative answer [S. V. Ivanov (1989)]

If $n > 10^{10}$ and $p > 5000$ is a prime, the word

$$w = [[x^{pn}, y^{pn}]^n, y^{pn}]^n$$

is not concise.

There exists G such that $G_w = \{w_1 = 1, w_2\}$ and $w(G) = \langle w_2 \rangle$ is infinite.

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Nevertheless, there are positive results in some classes of groups:

Definition

A word w is said to be *concise in a class $\mathcal{C}$ of groups* if, for every group $G \in \mathcal{C}$, $w(G)$ is finite whenever G_w is finite.

Consequence of Dicman's Lemma

Every word is concise in the class of periodic groups.

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Proof: Let G be a periodic group an w be an arbitrary word.
Put $x = w(g_1, \ldots, g_n) \in G_w$. Then $x^g \in G_w$, $\forall g \in G$.
It follows $|G : C_G(x)|$ finite and $|G : C_G(w(G))|$ finite, as well.
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Merzlyakov (1967)

Every word is concise in the class of linear groups.

R. F. Turner-Smith (1966)

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Groups in which $x^n = 1$ for all $x \in G$ are called *groups of finite exponent* (dividing n).

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Is every power of a multilinear word concise in the class of residually finite groups?

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Is every power of a multilinear word concise in the class of residually finite groups?

Theorem [C. Acciarri, P. Shumyatsky (2014)]

If w is a multilinear commutator and q is a prime-power, then the word w^q is concise in the class of residually finite groups.

It is unknown if w^q is concise (in the class of all groups).

Question

Does the result hold if q is allowed to be an arbitrary integer?

The above question seems really hard because the Hall-Higman theory does not work here.

Engel version of the Restricted Burnside Problem

Is every finitely generated n-Engel group nilpotent?

Groups in which $[x,_n y] = 1$ for all $x, y \in G$ are called *n-Engel*.

Positive solution (Havas, Vaughan-Lee) ... for $n \leq 4$.

A. Abdollahi and F. Russo - 2011 (G. A. Fernández-Alcober, M. Morigi and G. Traustason - 2012)

The n-th Engel word $[x, {}_n y]$ is concise for $n \leq 4$.

Positive solution for residually finite groups [Wilson]

Finitely generated residually finite n-Engel groups are nilpotent.

Similarly

The n-th Engel word $[x,_n y]$ is concise in res. fin. groups for any n.

Theorem [E. Detomi, M. Morigi, P. Shumyatsky (2017)]

Let w be a multilinear commutator.
Then $[w,_n y]$ is concise in residually finite groups, for any n.

Open problem

Is the word $[y,_n w]$ concise in residually finite groups, for any n?

YES... for $w = [x_1, \ldots, x_k]^q$ with k, n, q positive integers.

Is every power of a concise word concise?

Theorem [C. Acciarri, P. Shumyatsky (2014)]

If w is a multilinear commutator and q is a prime-power, then the word w^q is concise in the class of residually finite groups.

Claim [S. V. Ivanov's paper]

If $m = 3n$ is an odd integer ≥ 1005, then the word $[x^n, y^n]^m$ is not concise.

Is $[x^n, y^n]$ a concise word?

Tongue twister

Non-commutator words are concise.
Is a commutator of two non-commutator words a concise word?

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Theorem [C. Delizia, P. Shumyatsky, A. Tortora and M.T. (2019)]

For any non-commutator words $w_1 = w_1(x_1, \dots, x_r)$ and $w_2 = w_2(y_1, \dots, y_s)$, the word $[w_1, w_2]$ is concise.

Corollary

The word $[x^n, y^n]$ is concise for each n.

Claim [S. V. Ivanov's paper]

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Corollary

The word $[x^n, y^n]$ is concise for each n.

Claim [S. V. Ivanov's paper]

If $m = 3n$ is an odd integer ≥ 1005, then the word $[x^n, y^n]^m$ is not concise.

If the claim is correct, a power of a concise word needs not to be concise.

C. Delizia, P. Shumyatsky, A. Tortora and M.T., *On conciseness of some commutator words*, Arch. Math., vol. 112 (1) (2019), 27–32, https://doi.org/10.1007/s00013-018-1215-8.

Definition

A word w is *boundedly concise in a class $\mathcal{C}$* of groups if for every integer m there exists a number $v = v(\mathcal{C}, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in \mathcal{C}$ it always follows that $|w(G)| \leq v$.

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G. A. Fernández-Alcober, M. Morigi

Every word which is concise in the class of all groups is actually boundedly concise.

Conjecture (G. A. Fernández-Alcober, P. Shumyatsky)

Each word that is concise in residually finite groups is boundedly concise.

TRUE (E. Detomi, M. Morigi, P. Shumyatsky, 2019) ... except:

$[w,_n y]$ with $n \geq 1$ and $w = w(x_1, ..., x_k)$ a multilinear commutator word.

A *profinite group* is a topological group that is isomorphic to the inverse limit of an inverse system of discrete finite groups.

In the context of profinite groups all the usual concepts of group theory are interpreted topologically. In particular, by a subgroup of a profinite group we mean a closed subgroup. A subgroup is said to be generated by a set S if it is topologically generated by S. In particular, $w(G)$ is the minimal closed subgroup containing all the values of w.

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Jaikin-Zapirain:
Is every word concise in the class of profinite groups?

Every word is concise in profinite groups iff it is concise in res. fin. groups.

Conjecture [E. Detomi, M. Morigi, P. Shumyatsky (2016)]

Every word is concise in profinite groups in the following stronger sense:

If w is a word and G a profinite group such that $|G_w| \leq \aleph_0$, then $w(G)$ is finite.

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Theorem [E. Detomi, M. Morigi, P. Shumyatsky (2016)]

Let G be a profinite group and w be one of the following words:
w a multilinear word,
$w = x^2$,
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If G has only countably many w-values, then $w(G)$ is finite.

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Open cases

Power words and Engel words!

E. Detomi, M. Morigi, P. Shumyatsky, *On conciseness of words in profinite groups*, J. Pure Appl. Algebra, vol. 220 (2016), 3010–3015.

Definition [E. Detomi, B. Klopsch, P. Shumyatsky (2019)]

A word w is said to be *strongly concise in a class $\mathcal{C}$ of profinite groups* if, for every group G in $\mathcal{C}$, $w(G)$ is finite whenever G_w has cardinality less than $2^{\aleph_0}$.

A word w is said to be *strongly concise* if it is strongly concise in the class of all profinite groups.

Theorem [E. Detomi, B. Klopsch, P. Shumyatsky (2019)]

Every multilinear word is strongly concise.

Theorem [E. Detomi, B. Klopsch, P. Shumyatsky (2019)]

Every word is strongly concise in the class of nilpotent profinite groups.

Theorem [E. Detomi, B. Klopsch, P. Shumyatsky (2019)]

The following group words are strongly concise:

$$x^2, \quad x^3, \quad x^6,$$
$$[x^2, y], \quad [x^3, y], \quad [x, y, y],$$
$$[x^2, z_1, \dots, z_r], \quad [x^3, z_1, \dots, z_r], \quad [x, y, y, z_1, \dots, z_r], \quad r \geq 1.$$

E. Detomi, B. Klopsch, P. Shumyatsky, *Strong conciseness in profinite groups* , arXiv:1907.01344v1 [math.GR] (2019).

THANK YOU!