Tensor Products of Restricted Simples of SL_4 over Characteristic 2

R. A. Spencer

DPMMS, University of Cambridge

The Question

If $\{L(\lambda)\ :\ \lambda \in \Lambda\}$ is the set of all **simple** SL_4 modules over an algebraically closed field $\mathbb{k}$ of characteristic 2, **what is the structure** of $L(\lambda) \otimes_{\mathbb{k}} L(\mu)$?

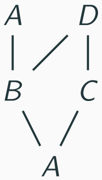
A Generalised Form of Alperin Diagram

Alperin Diagrams: An Overview

- Diagram for conveying **submodule structure**
- Defined in the 1980s, but often used **loosely**
- Only describes a **small** class of modules

Definition (Often)

An Alperin Diagram for module M is a **quiver** Q with a **lattice bijection** δ from the lattice of **arrow closed subsets of** Q to the **lattice of submodules of** M.



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- Edges correspond to **non-split extensions as subquotients**

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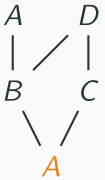

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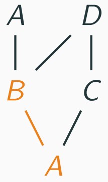

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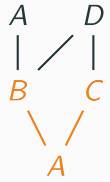

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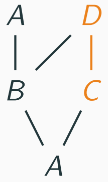

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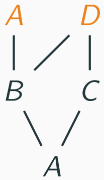

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Alperin Diagrams: When They Fail and Alternatives

Problems:

- The requirement δ is a **bijection** is very strong
- Requires **infinite quivers** or only **finitely many submodules**
- Infinitely many submodules occur **frequently** (e.g. $\mathbb{R} \oplus \mathbb{R}$ over $\mathbb{R}$)

Possible Solutions:

- **Drop surjectivity** requirement on δ
  - Generalise diagrams based on certain classes of filtrations (e.g. radical, socle, socle-isotypic, etc.)
  - Require **socle** and **radical** series to be read off

Alperin Diagrams: Our Alternative

- An **injective** diagram, based on **generated submodules**, annotated to give the **socle and radical** series
- Procedure for module M:
  - Find n vectors $\{v_i\}$ where n is the **composition length** of M such that,
    - $\langle v_1 \rangle = M$
    - $\langle v_i \rangle = \langle v_j \rangle \iff i = j$
    - $\langle v_i \rangle / \operatorname{rad} \langle v_i \rangle$ is **simple**
  - Draw a line $v_i \to v_j$ if $v_j \in \operatorname{rad} \langle v_i \rangle \backslash \operatorname{rad}^2 \langle v_i \rangle$ and $\langle v_j \rangle / \operatorname{rad}^2 \langle v_i \rangle \hookrightarrow \langle v_i \rangle / \operatorname{rad}^2 \langle v_i \rangle$ is **not split**
  - Construct δ to take the **arrow-closure** of v_i to $\langle v_i \rangle$, be **lattice** and **top** preserving.
- **Decorate** with more vectors to highlight **socle** and **radical series** and other **submodule structure**.
- Examples to come in the context of quasi-hereditary algebras

Quasi-Hereditary Algebras

Quasi-hereditary Algebras

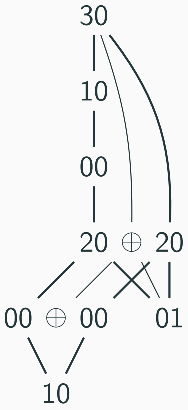
- Really a class of categories of **modules**
- **Simple** modules $L(\lambda)$ labeled by poset $(\Lambda, \leq)$
- **Standard** and **costandard** modules $\Delta(\lambda)$ and $\nabla(\lambda)$ for each $\lambda \in \Lambda$
  - Simple head (resp. socle) of $L(\lambda)$
  - All other factors $L(\mu)$ for $\mu < \lambda$
  - Maximal such quotient of projective cover (resp. submodule of injective hull) of $L(\lambda)$
- Indecomposable **tilting** modules (both Δ- and ∇-filtrations) $T(\lambda)$

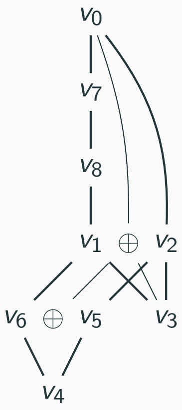
Rational (co)Modules of Algebraic Groups

- Λ is the set of dominant weights
    - **Tuples of naturals**
- $\leq$ not lexicographical: depends on certain **coroots**
- Each $L(\lambda)$, $\Delta(\lambda)$, $\nabla(\lambda)$ and $T(\lambda)$ have highest weight λ.
- Contravariant dual
    - Tilting modules **contravariantly self-dual**

Example of Alternative Alperin Diagram

The module $\Delta(3,0)$ of type G_2 over characteristic 2





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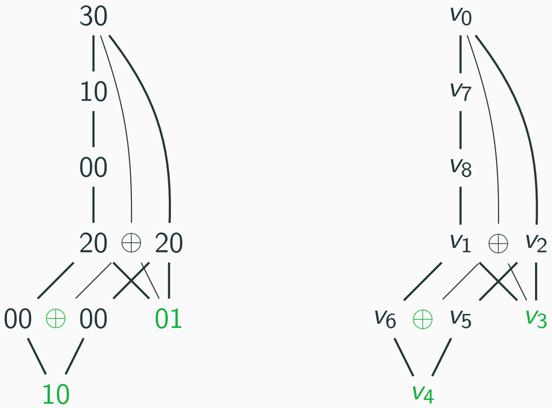

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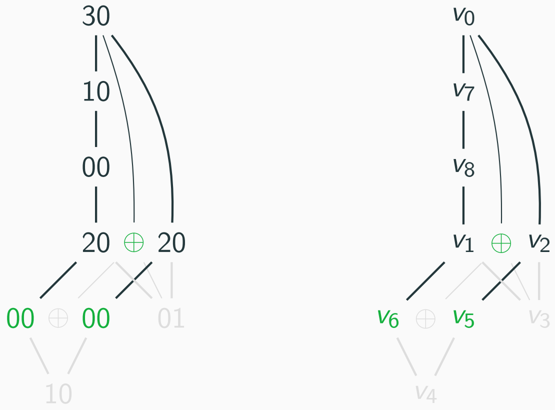

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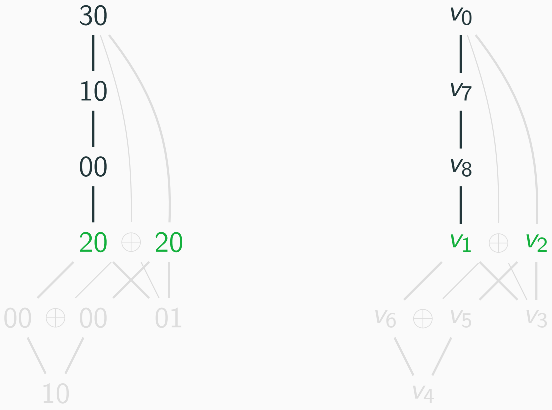

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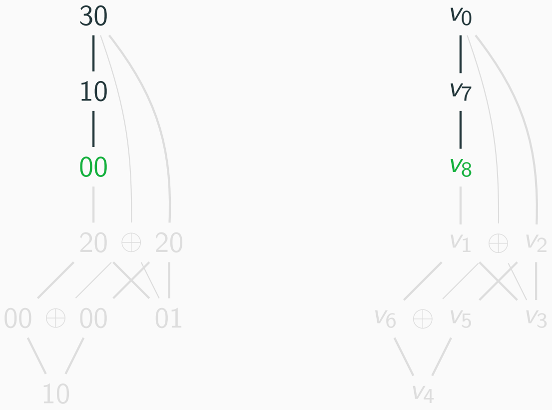

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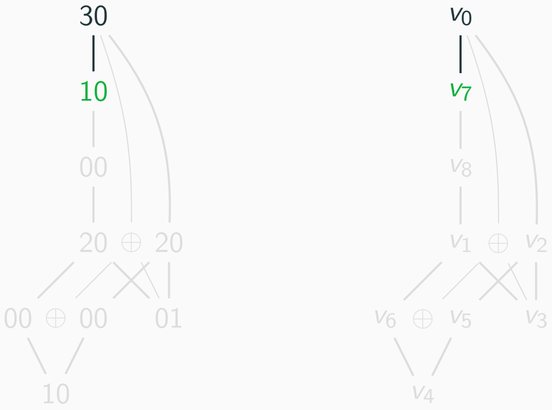

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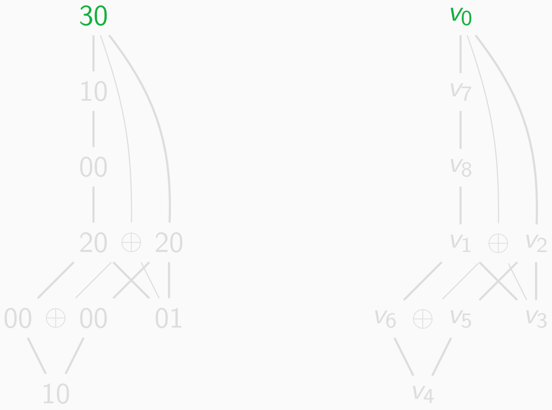

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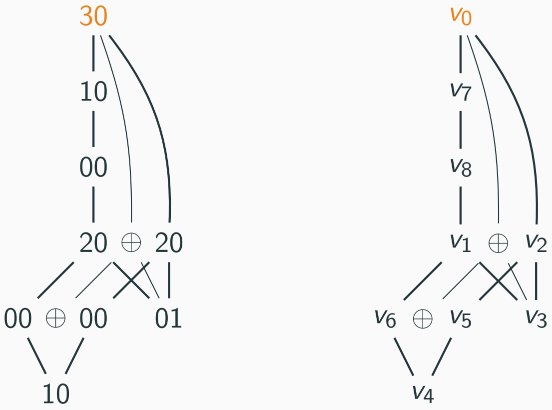

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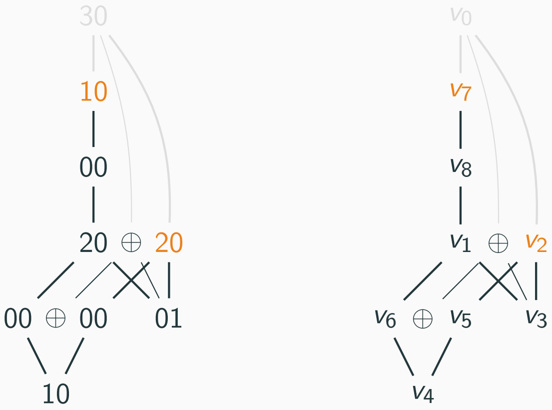

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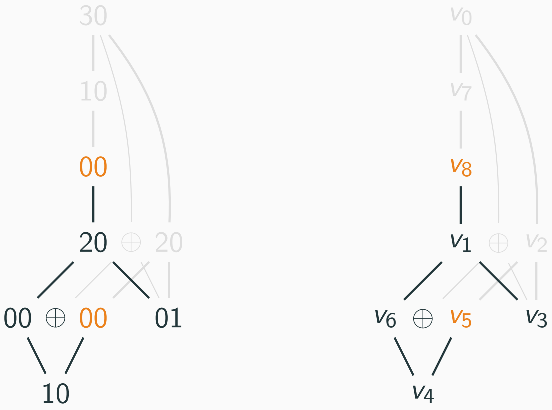

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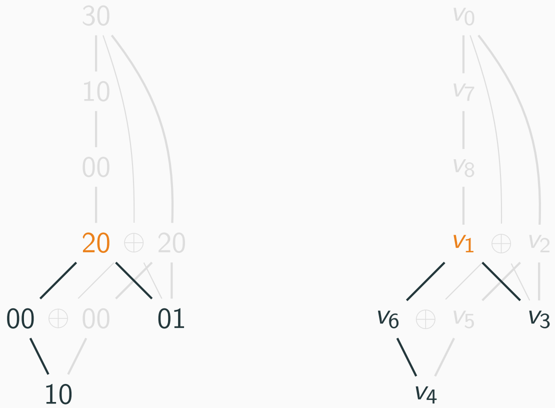

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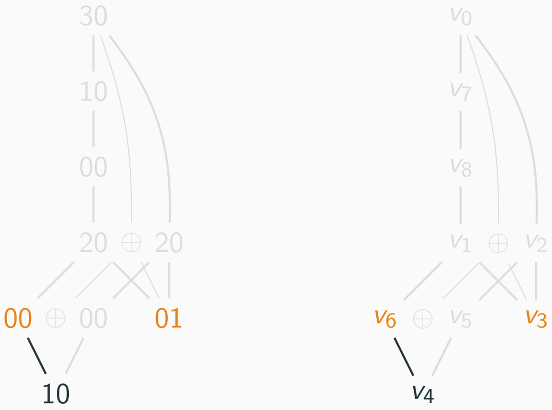

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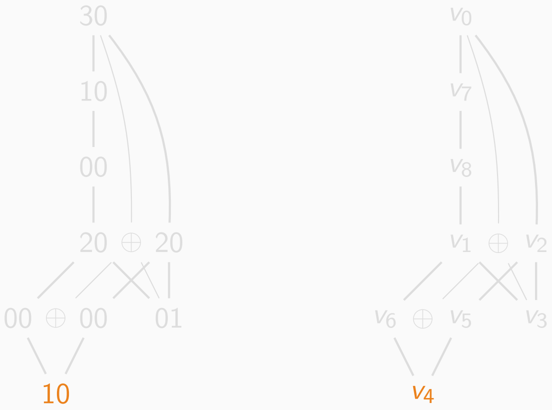

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Tensor Products of Simples of SL_4

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Philosophy

- "Twisting" by the **Frobenius automorphism** of G allows us to reduce to finitely many cases sometimes
- Write "base p"

$$\lambda = \sum_{j \geq 0} p^j \lambda_j \quad , \quad \mu = \sum_{j \geq 0} p^j \mu_j$$

  for p-**restricted** weights λ_j and μ_j
  - E.g.
    $(3, 14, 5) = (1, 0, 1) + 2 \times (1, 1, 0) + 2^2 \times (0, 1, 1) + 2^3 \times (0, 1, 0)$
- By the **Steinburg tensor product theorem**

$$L(\lambda) \otimes L(\mu) \cong \bigotimes_{j \in \mathbb{N}_0} \left(L(\lambda_j) \otimes L(\mu_j)\right)^{[j]}$$

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Helpful Facts

- Some restricted $L(\lambda) = \nabla(\lambda) = \Delta(\lambda) = T(\lambda)$
- Tiling modules are closed under $\otimes$
- In some cases, software can give form of $\Delta(\lambda)$ (and $\nabla(\lambda)$)
- Structure of contravariant dual can be read off (halving the amount of work)
- Simple modules divide up into blocks

Example: SL_4 over characteristic 2

- 2-restricted weights are elements of $\{0, 1\}^3$
- Only cases not covered by symmetry (or trivial) are

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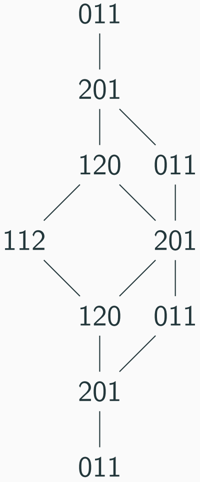
- The others are contravariantly self dual but not tilting

Example of Calculation: $001 \otimes 111$

- **Characters** gives composition factors with multiplicites: $011^4, 112, 120^2, 201^3$ in one block



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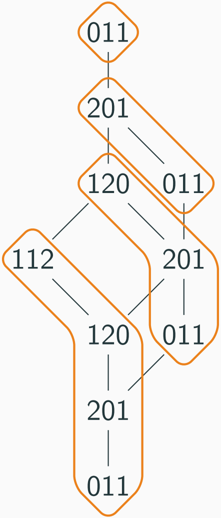

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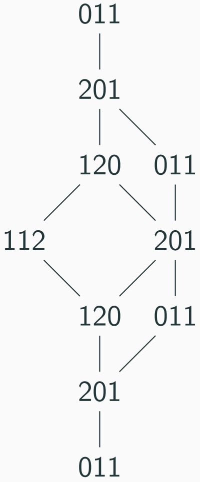

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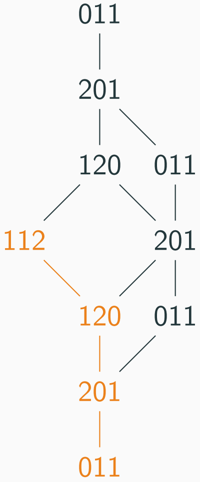

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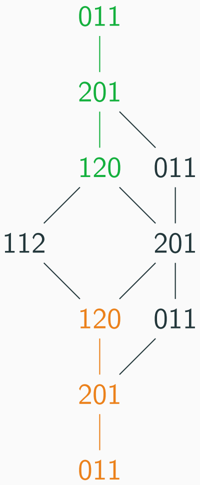

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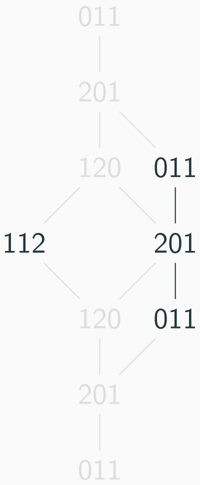

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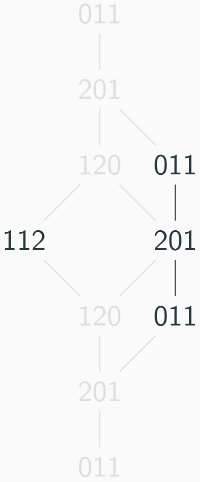

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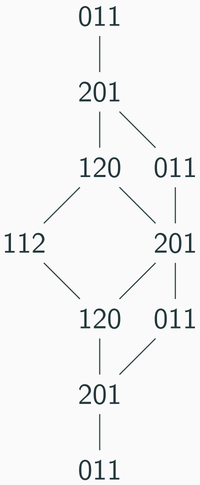

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- Can give structure of all indecomposable summands

Application

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- **False** (Bendel, Nakano, Pillen, and Sobaje '19) for type G_2 over characteristic 2

Donkin's Tilting Module Conjecture for SL_4 over characteristic 2

Theorem (Sobaje '18)

Donkin's conjecture holds for G iff

$$\left(L(\rho) \otimes L(\rho)\right)^{\oplus p^{rank G}} \cong \bigoplus_{\lambda \in X_1} T\left((p-1)\rho + \lambda\right) \otimes L\left((p-1)\rho - \lambda\right)$$

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Corollary

Donkin's conjecture holds for type A_3 in characteristic 2.

Thank You

Questions?