On analogues of R. Baer's results on the hypercenter of finite groups in the formation theory

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The Hypercenter

All considered groups are finite.

1. The notion of the hypercenter naturally appears with the definition of a nilpotent group through the upper central series.
2. As it was shown by Burnside a group is equal to its hypercenter iff all its Sylow subgroups are normal.
3. One of the first characterizations of the hypercenter as the intersection of some system of subgroups were obtained by Hall in 1937.
4. The previous result was generalized by Baer in 1953.

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The $\mathfrak{F}$-hypercenter

1. In 1959 Baer suggested the analogue of the hypercenter for supersoluble groups.
2. In 1968 Huppert extended the notion of hypercenter for a local formation $\mathfrak{F}$ with the help of a local definition of $\mathfrak{F}$.
3. In 1974 Shemetkov extended the notion of hypercenter for a graduated formation.
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Let $\mathfrak{X}$ be a class of groups. A chief factor H/K of a group G is called $\mathfrak{X}$-central if

$$(H/K) \rtimes G/C_G(H/K) \in \mathfrak{X},$$

otherwise it is called $\mathfrak{X}$-eccentric.

A normal subgroup N of G is said to be $\mathfrak{X}$-hypercentral in G if $N = 1$ or $N \neq 1$ and every chief factor of G below N is $\mathfrak{X}$-central.

The $\mathfrak{X}$-hypercenter $\mathrm{Z}_{\mathfrak{X}}(G)$ is the product of all normal $\mathfrak{X}$-hypercentral subgroups of G.

If $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups then $\mathrm{Z}_{\mathfrak{N}}(G)$ is just the hypercenter $\mathrm{Z}_{\infty}(G)$ of a group G.

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Shemetkov's problem

Recall that a subgroup U of G is called $\mathfrak{X}$-*maximal* in G provided that (a) $U \in \mathfrak{X}$; and (b) if $U \leq V \leq G$ and $V \in \mathfrak{X}$, then $U = V$.

The symbol $\mathrm{Int}_{\mathfrak{X}}(G)$ denotes the intersection of all $\mathfrak{X}$-maximal subgroups of G.

It is well known that the intersection of maximal abelian subgroups of G is the center of G. In 1953 Baer showed that the intersection of maximal nilpotent subgroups of G is the hypercenter of G.
The intersection of maximal supersoluble subgroups of G does not necessary coincide with the supersoluble hypercenter of G.

Shemetkov possed the following problem at Gomel Algebraic Seminar in 1995: "***For what non-empty normally hereditary solubly saturated formations*** $\mathfrak{X}$ ***does the equality*** $\mathrm{Int}_{\mathfrak{X}}(G) = \mathrm{Z}_{\mathfrak{X}}(G)$ ***hold for every group*** G***?***"

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Formations

A *formation* is a class $\mathfrak{X}$ of groups with the following properties:
(a) every homomorphic image of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group;
(b) if G/M and G/N are $\mathfrak{X}$-groups, then also $G/(M \cap N) \in \mathfrak{X}$.

A formation $\mathfrak{X}$ is said to be:
(a) *saturated* (respectively *solubly saturated*) if $G \in \mathfrak{X}$ whenever $G/\Phi(N) \in \mathfrak{X}$ for some normal (respectively for some soluble normal) subgroup N of G;

(b) *hereditary* (respectively *normally hereditary*) if $H \in \mathfrak{X}$ whenever $H \leq G \in \mathfrak{X}$ (respectively whenever $H \trianglelefteq G \in \mathfrak{X}$).

1. If $\mathfrak{F}$ is a formation and $G \in \mathfrak{F}$, then $G = \mathrm{Z}_{\mathfrak{F}}(G)$.
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Local Formations

Recall that a function of the form $f : \mathbb{P} \to \{formations\}$ is called a *formation function* and a formation $\mathfrak{F}$ is called *local* if

$$\mathfrak{F} = (G \mid G/C_G(\overline{H}) \in f(p) \text{ for every } p \in \pi(\overline{H}) \text{ and every chief factor } \overline{H} \text{ of } G)$$

for a formation function f. In this case f is called a *local definition* of $\mathfrak{F}$.

By the Gaschütz-Lubeseder-Schmid theorem, a formation is local if and only if it is non-empty and *saturated*

If $\mathfrak{F}$ is a local formation, there exists a unique formation function F, defining $\mathfrak{F}$, such that

$$F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F} \text{ for every } p \in \mathbb{P}.$$

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The solution of this problem for hereditary saturated formation were obtained Skiba (for the soluble case, see also Beidleman and Heineken).

Theorem (Skiba)

Let F be the canonical local definition of a hereditary saturated formation $\mathfrak{F}$ and $\pi(\mathfrak{F}) \neq \emptyset$. The $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group G if and only if for every prime p formation $\mathfrak{F}$ contains every group whose maximal subgroups belong $F(p)$.

It is necessary to note that the methods of these papers are not applicable for non-saturated or non-hereditary formations.

Thus, the answer to the Shemetkov's question was not known even in such an important special case, when $\mathfrak{X} = \mathfrak{N}^*$ is the class of all quasinilpotent groups.

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Recall that G is called a *quasi-$\mathfrak{F}$-group* if for every $\mathfrak{F}$-eccentric chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K. We use $\mathfrak{F}^*$ to denote the class of all quasi-$\mathfrak{F}$-groups.
If $\mathfrak{N} \subseteq \mathfrak{F}$ is a normally hereditary saturated formation, then $\mathfrak{F}^*$ is a normally hereditary Baer-local (solubly saturated) formation.

Theorem

Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups. Then $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group G if and only if $\mathrm{Int}_{\mathfrak{F}^*}(G) = \mathrm{Z}_{\mathfrak{F}^*}(G)$ holds for every group G.

Corollary

The intersection of all maximal quasinilpotent subgroups of a group is its quasinilpotent hypercenter.

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Example

Let $\mathfrak{N}_{ca}$ be a class of groups whose abelian chief factors are central and non-abelian are simple groups.

Let $\mathfrak{J}$ be a class of simple groups with nilpotent outer automorphism group and $\mathfrak{N}_{\mathfrak{J}}$ be a class of groups whose abelian chief factors are central and non-abelian are simple groups from $\mathfrak{J}$.

The classes $\mathfrak{N}_{ca}$ and $\mathfrak{N}_{\mathfrak{J}}$ are normally hereditary solubly saturated formations.

Then

$$\mathrm{Int}_{\mathfrak{N}_{\mathfrak{J}}}(G) = \mathrm{Z}_{\mathfrak{N}_{\mathfrak{J}}}(G)$$

holds for every group G and there is a group H with

$$\mathrm{Int}_{\mathfrak{N}_{ca}}(G) \neq \mathrm{Z}_{\mathfrak{N}_{ca}}(G)$$

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The classes $\mathfrak{N}_{ca}$ and $\mathfrak{N}_{\mathfrak{J}}$ are normally hereditary solubly saturated formations.

Then

$$\mathrm{Int}_{\mathfrak{N}_{\mathfrak{J}}}(G) = \mathrm{Z}_{\mathfrak{N}_{\mathfrak{J}}}(G)$$

holds for every group G and there is a group H with

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The Direct Product of Formations

Recall that $\mathfrak{G}_\pi$ is the class of all π-groups where π is a set of primes. The class

$$\bigtimes_{i \in I} \mathfrak{F}_{\pi_i} = (G \mid \; \mathrm{O}_{\pi_i}(G) \in \mathfrak{F}_{\pi_i} \text{ is a Hall } \pi_i\text{-subgroup of } G)$$

is a hereditary saturated formation where $\sigma = \{\pi_i | i \in I\}$ is a partition of $\mathbb{P}$ into mutually disjoint subsets and $\mathfrak{F}_{\pi_i}$ is a hereditary saturated formation with $\pi(\mathfrak{F}_{\pi_i}) = \pi_i$ for all $i \in I$.

Theorem

Assume that $\sigma = \{\pi_i | i \in I\}$ *is a partition of* $\mathbb{P}$ *into mutually disjoint subsets and* $\mathfrak{F} = \underset{i \in I}{\times} \mathfrak{G}_{\pi_i}$. *The following properties of a* π_i*-element* g *in* G *are equivalent:*

(1) $g \in \mathrm{Z}_{\mathfrak{F}}(G)$;

(2) $gx = xg$ *for all* π_i'*-elements* x *of* G;

(3) $|G : C_G(\langle g \rangle^G)|$ *is a* π_i*-number;*

(4) $|G : C_G(g)|$ *is a* π_i*-number and* $G^{\mathfrak{F}} \leq C_G(g)$.

Corollary (R. Baer, 1953)

The following properties for a p-element g *of a group* G *are equivalent:*

(1) $g \in \mathrm{Z}_{\infty}(G)$;

(2) g *permutes with every* p'*-element of* G;

(3) $|G : C_G(g^G)|$ *is a power of* p;

(4) $|G : C_G(g)|$ *is a power of* p *and* $G^{\mathfrak{N}} \leq C_G(g)$.

Remarks

V. I. Murashka, On one generalization of Baer's theorems on hypercenter and nilpotent residual. Prob. Fiz. Mat. Tech., **16** (2013) 84–88.

Every formation $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$ is a lattice formation, i.e. formation for which the set of all $\mathfrak{F}$-subnormal subgroups of every group G is a sublattice of the subgroup lattice of G.

A. Ballester-Bolinches, L. M. Ezquerro, Classes of Finite groups, Springer, 2006.

Skiba extended the theory of nilpotent groups on such formations (theory of σ-nilpotent groups).

A. N. Skiba, On σ-subnormal and σ-permutable subgroups of finite groups. J. Algebra **436** (2015) 1–16.

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The Intersection of Normalizers of $\mathfrak{F}$-maximal Subgroups

H. Wielandt, Zusammenghesetzte Gruppen: Hölder Programm heute. The Santa Cruz conf. on finite groups, Santa Cruz, 1979. Proc. Sympos. Pure Math. 37, Providence RI: Amer. Math. Soc. (1980) 161–173.

Denote the intersection of all normalizers of $\mathfrak{F}$-maximal subgroups of G by $\mathrm{NI}_{\mathfrak{F}}(G)$.

Theorem

Let $\sigma = \{\pi_i | i \in I\}$ be a partition of $\mathbb{P}$ into mutually disjoint subsets, $\mathfrak{F}_{\pi_i}$ be a hereditary saturated formation with $\pi(\mathfrak{F}_{\pi_i}) = \pi_i$ for all $i \in I$ and $\mathfrak{F} = \times_{i \in I} \mathfrak{F}_{\pi_i}$. The following statements are equivalent:

(1) $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *for every group* G*;*

(2) $\mathrm{Int}_{\mathfrak{F}_{\pi_i}}(G) = \mathrm{Z}_{\mathfrak{F}_{\pi_i}}(G)$ *for every* π_i*-group* G *and every* $i \in I$*;*

(3) $\bigcap_{i \in I} \mathrm{NI}_{\mathfrak{F}_{\pi_i}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *for every group* G*.*

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Corollaries

Corollary

Let G be a group. Then
(1) (Hall) The hypercenter of G is the intersection of all normalizers of all Sylow subgroups of G.
(2) (Baer) The hypercenter of G is the intersection of all maximal nilpotent subgroups of G.

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(1) The intersection of all normalizers of all π_i-maximal subgroups of G for all $i \in I$ is the $\mathfrak{F}$-hypercenter of G.
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(2) (Skiba) The intersection of all $\mathfrak{F}$-maximal subgroups of G is the $\mathfrak{F}$-hypercenter of G.

Recall that G *satisfies the Sylow tower property* if G has a normal series whose factors are isomorphic to Sylow subgroups of G.

Theorem

Let $\mathfrak{F}$ be a hereditary saturated formation, F be its canonical local definition and N be a normal subgroup of G that satisfies the Sylow tower property. Then $N \leq \mathrm{Z}_{\mathfrak{F}}(G)$ if and only if $N_G(P)/C_G(P) \in F(p)$ for all $P \in \mathrm{Syl}_p(N)$ and $p \in \pi(N)$.

Supersoluble-like Classes of Groups

Recall that a subgroup H of a group G is called $\mathbb{P}$-subnormal if either $H = G$ or there exists a chain of subgroups $H = H_0 < \cdots < H_n = G$ such that $|H_i : H_{i-1}|$ is a prime number.

Recall that a group G is called widely w-supersoluble (resp., v-supersoluble) if all Sylow subgroups (resp. cyclic primary) of G are $\mathbb{P}$-subnormal in G.

Like the class of all supersoluble groups $\mathfrak{U}$ the classes of all w-supersoluble groups $\mathrm{w}\mathfrak{U}$ and v-supersoluble groups $\mathrm{v}\mathfrak{U}$ are a hereditary saturated formations with the Sylow tower of supersoluble type.

A. F. Vasil'ev, T. I. Vasil'eva and V. N. Tyutyanov, On the finite groups of supersoluble type. Sib. Math. J. **51**(6) (2010) 1004–1012.

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Corollary

Let N be a normal subgroup of a group G with the Sylow tower of supersoluble type.

(a) *(Baer)* $N \leq \mathrm{Z}_{\mathfrak{U}}(G)$ *if and only if*

$$N_G(P)/C_G(P) \in \mathfrak{N}_p\mathfrak{A}(p-1) \text{ for all } P \in \mathrm{Syl}_p(N) \text{ and } p \in \pi(N).$$

(b) $N \leq \mathrm{Z}_{w\mathfrak{U}}(G)$, *if and only if*

$$N_G(P)/C_G(P) \in \mathfrak{N}_p(\mathcal{A}(p-1) \cap w\mathfrak{U}) \text{ for all } P \in \mathrm{Syl}_p(N) \text{ and } p \in \pi(N).$$

(d) $N \leq \mathrm{Z}_{v\mathfrak{U}}(G)$ *if and only if*

$$N_G(P)/C_G(P) \in \mathfrak{N}_p(\mathfrak{S}(p-1) \cap v\mathfrak{U}) \text{ for all } P \in \mathrm{Syl}_p(N) \text{ and } p \in \pi(N).$$

V. I. Murashka, Properties of the class of finite groups with P-subnormal cyclic primary subgroups // Dokl. NAN Belarusi. **58**(1) (2014) 5–8 (In Russian).

V. I. Murashka, On analogues of Baer's theorems for widely supersoluble hypercenter of finite groups. Asian-European J. Math. **11**(3) (2018) 1850043 (8 pages).

Baer-local (Composittion) Formations

Recall that a function of the form $f : \mathbb{P} \cup \{0\} \to \{\textit{formations}\}$ is called a composition definition. A formation $\mathfrak{F}$ is called *composition* or *Baer-local* if

$$\mathfrak{F} = (G \mid G/G_{\mathfrak{S}} \in f(0) \text{ and } G/C_G(\overline{H}) \in f(p) \text{ for every abelian } p\text{-chief factor } \overline{H} \text{ of } G)$$

for some composition definition f.

A formation is composition (Baer-local) if and only if it is *solubly saturated*.

Recall that any nonempty composition formation $\mathfrak{F}$ has an unique composition definition F such that $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$ for all primes p and $F(0) = \mathfrak{F}$. In this case F is called the *canonical composition definition* of $\mathfrak{F}$.

Every non-empty composition formation $\mathfrak{F}$ contains the greatest by inclusion local subformation $\mathfrak{F}_l$.

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Reduction Theorem

Theorem

Let F be the canonical composition definition of a non-empty solubly saturated formation $\mathfrak{F}$. Assume that $F(p) \subseteq \mathfrak{F}_l$ for all $p \in \mathbb{P}$ and $\mathfrak{F}_l$ is hereditary.

(1) *Assume that* $\mathrm{Int}_{\mathfrak{F}_l}(G) = \mathrm{Z}_{\mathfrak{F}_l}(G)$ *holds for every group G. Let*

$$\mathfrak{H} = (S \text{ is a simple group} \mid \text{every } \mathfrak{F}\text{-central chief } D_0(S)\text{-factor is } \mathfrak{F}_l\text{-central}).$$

Then every chief $D_0\mathfrak{H}$-factor of G below $\mathrm{Int}_{\mathfrak{F}}(G)$ *is $\mathfrak{F}_l$-central in G.*

(2) *If* $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *holds for every group G, then* $\mathrm{Int}_{\mathfrak{F}_l}(G) = \mathrm{Z}_{\mathfrak{F}_l}(G)$ *holds for every group G.*

V. I. Murashka, On one question of Shemetkov about composition formations. arXiv:1904.04244v1 [math.GR] 7 Apr 2019

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Rank formations

Let $\overline{N}$ be a chief factor of G. Then $\overline{N} = \overline{N}_1 \times \cdots \times \overline{N}_n$ where $\overline{N}_i$ are isomorphic simple groups.
The number $n = r(\overline{N}, G)$ is the *rank* of $\overline{N}$ in G.
A *rank function* R is a map which associates with each prime p a set $R(p)$ of natural numbers. For each rank function let

$$\mathfrak{E}(R) = (G \in \mathfrak{S} \mid \text{for all } p \in \mathbb{P} \text{ each } p\text{-chief factor of } G \text{ has rank in } R(p)).$$

H. Heineken, Group classes defined by chief factor ranks. Boll. Un. Mat. Ital. B. **16** (1979) 754–764.
D. Harman, Characterizations of some classes of finite soluble groups. Ph.D. thesis, University of Warwick, 1981.
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Definition

(1) A generalized rank function $\mathcal{R}$ is a map defined on direct products of isomorphic simple groups by
(a) $\mathcal{R}$ associates with each simple group S a pair $\mathcal{R}(S) = (A_{\mathcal{R}}(S), B_{\mathcal{R}}(S))$ of possibly empty disjoint sets $A_{\mathcal{R}}(S)$ and $B_{\mathcal{R}}(S)$ of natural numbers.
(b) If N is the direct products of simple isomorphic to S groups, then $\mathcal{R}(N) = \mathcal{R}(S)$.

(2) Let $\overline{N}$ be a chief factor of G. We shall say that a *generalized rank* of $\overline{N}$ in G lies in $\mathcal{R}(\overline{N})$ (briefly $gr(\overline{N}, G) \in \mathcal{R}(\overline{N})$) if $r(\overline{N}, G) \in A_{\mathcal{R}}(\overline{N})$ or $r(\overline{N}, G) \in B_{\mathcal{R}}(\overline{N})$ and if some $x \in G$ fixes a composition factor $\overline{H}/\overline{K}$ of $\overline{N}$ (i.e. $\overline{H}^x = \overline{H}$ and $\overline{K}^x = \overline{K}$), then x induces an inner automorphism on it.

(3) With each generalized rank function $\mathcal{R}$ and a class of groups $\mathfrak{X}$ we associate a class

$$\mathfrak{X}(\mathcal{R}) = (G \mid \overline{H} \notin \mathfrak{X} \text{ and } gr(\overline{H}, G) \in \mathcal{R}(\overline{H}) \text{ for every } \mathfrak{X}\text{-eccentric chief factor } \overline{H} \text{ of } G)$$

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(2) Let $\overline{N}$ be a chief factor of G. We shall say that a *generalized rank* of $\overline{N}$ in G lies in $\mathcal{R}(\overline{N})$ (briefly $gr(\overline{N}, G) \in \mathcal{R}(\overline{N})$) if $r(\overline{N}, G) \in A_{\mathcal{R}}(\overline{N})$ or $r(\overline{N}, G) \in B_{\mathcal{R}}(\overline{N})$ and if some $x \in G$ fixes a composition factor $\overline{H}/\overline{K}$ of $\overline{N}$ (i.e. $\overline{H}^x = \overline{H}$ and $\overline{K}^x = \overline{K}$), then x induces an inner automorphism on it.

(3) With each generalized rank function $\mathcal{R}$ and a class of groups $\mathfrak{X}$ we associate a class

$$\mathfrak{X}(\mathcal{R}) = (G \mid \overline{H} \notin \mathfrak{X} \text{ and } gr(\overline{H}, G) \in \mathcal{R}(\overline{H}) \text{ for every } \mathfrak{X}\text{-eccentric chief factor } \overline{H} \text{ of } G)$$

Definition

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1. Let $\mathfrak{E} = (1)$. Assume that $\mathcal{R}(H) = (\{1\}, \emptyset)$ if H is abelian and $\mathcal{R}(H) = (\emptyset, \emptyset)$ otherwise. Then $\mathfrak{E}(\mathcal{R}) = \mathfrak{U}$.
2. If $\mathcal{R}(H) \equiv (\{1\}, \emptyset)$, then $\mathfrak{E}(\mathcal{R})$ is the class $\mathfrak{U}_c$ of all c-supersoluble groups.
3. Let $\mathfrak{J}$ be a class of simple groups. If $\mathcal{R}(H) \equiv (\{1\}, \emptyset)$ for $H \in \mathfrak{J}$ and $\mathcal{R}(H) = (\mathbb{N}, \emptyset)$ otherwise, then $\mathfrak{E}(\mathcal{R})$ is the class of all $\mathfrak{J}c$-supersoluble groups.
4. Assume that $\mathcal{R}(H) = (A_{\mathcal{R}}(H), \emptyset)$ if H is abelian and $\mathcal{R}(H) = (\emptyset, \emptyset)$ otherwise. Then $\mathcal{R}$ is a rank function.
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We shall say that a generalized rank function $\mathcal{R}$ is (resp. *strongly*) *hereditary* if for any simple group S holds:
(a) from $a \in A_{\mathcal{R}}(S)$ it follows that $b \in A_{\mathcal{R}}(S)$ for any natural $b|a$ (resp. $b \leq a$);
(b) from $a \in B_{\mathcal{R}}(S)$ it follows that $b \in A_{\mathcal{R}}(S) \cup B_{\mathcal{R}}(S)$ (resp.$b \in B_{\mathcal{R}}(S)$) for any natural $b|a$ (resp. $b \leq a$).

Theorem

Let $\mathfrak{N} \subseteq \mathfrak{F}$ be a composition formation with the canonical composition definition F and $\mathcal{R}$ be a generalized rank function. Then
(1) $\mathfrak{F}(\mathcal{R})$ is a composition formation with the canonical composition definition $F_{\mathcal{R}}$ such that $F_{\mathcal{R}}(0) = \mathfrak{F}(\mathcal{R})$ and $F_{\mathcal{R}}(p) = F(p)$ for all $p \in \mathbb{P}$.
(2) If $\mathfrak{F}$ is normally hereditary and $\mathcal{R}$ is hereditary, then $\mathfrak{F}(\mathcal{R})$ is normally hereditary.

The Main Result

Theorem

Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups, m be a natural number with $\mathfrak{G}_{\{q \in \mathbb{P} \mid q \leq m\}} \subseteq \mathfrak{F}$, $\mathcal{R}$ be a strongly hereditary rank function such that $\mathcal{R}(N) \subseteq [0, m]$ for any simple group N. Then the following statements are equivalent:

(1) $\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ *holds for every group G and*

$$\bigcup_{n=1}^{m}(\mathrm{Out}(G) \wr S_n \mid G \notin \mathfrak{F} \text{ is a simple group and } n \in A_{\mathcal{R}}(G)) \subseteq \mathfrak{F}.$$

(2) $\mathrm{Z}_{\mathfrak{F}(\mathcal{R})}(G) = \mathrm{Int}_{\mathfrak{F}(\mathcal{R})}(G)$ *holds for every group G.*

Remark

Theorem

Let $\mathfrak{F} \neq \mathfrak{G}$ be a hereditary saturated formation containing all nilpotent groups and $\mathcal{R}$ be a strongly hereditary generalized rank function.
(1) *Assume that $\mathrm{Z}_{\mathfrak{F}(\mathcal{R})}(G) = \mathrm{Int}_{\mathfrak{F}(\mathcal{R})}(G)$ holds for every group G. Let*

$$C_1 = \min_{G \in \mathcal{M}(\mathfrak{F}) \textit{ with } \mathrm{F}(G) = \tilde{\mathrm{F}}(G)} \ \max_{M \textit{ is a maximal subgroup of } G} |M| - 1.$$

Then $\mathcal{R}(S) \subseteq [0, C_1]$ for every simple group $S \notin \mathfrak{F}$.
(2) *Let*

$$C_2 = \max\{m \in \mathbb{N} \mid \mathfrak{G}_{\{q \in \mathbb{P} \mid q \leq m\}} \subseteq \mathfrak{F}\}.$$

If $\mathcal{R}(S) \subseteq [0, C_2]$ for every simple group $S \notin \mathfrak{F}$, then $gr(\overline{H}, G) \in \mathcal{R}(\overline{H})$ for every G-composition factor $\overline{H} \notin \mathfrak{F}$ below $\mathrm{Int}_{\mathfrak{F}(\mathcal{R})}(G)$.

THANK YOU FOR ATTENTION