Characters in π-separable and p-solvable groups

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The B_π-characters

π-separable groups and Hall π-subgroups

Let π be a set of primes and denote as π' the complement set of π. A natural number n is called a π-number if all its prime divisors are in π. For any $m \in \mathbb{N}$, m_π is the π-*part* of m, i.e., the maximal π-number dividing m.

Definition

A finite group G is called π-separable if, given a composition series

$$G = N_0 \rhd N_1 \rhd \ldots \rhd N_r = \{1\},$$

then each factor group N_1/N_{i+1} is either a π-group or a π'-group, i.e., its order is either a π-number or a π'-number.

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Definition

A subgroup $H \leq G$ is called Hall π-subgroup of G if $|H| = |G|_\pi$.

If G is a π-separable group, then it has a Hall π-subgroup and two distinct Hall π-subgroups are conjugated.

The π-special characters

Definition

Let G be a finite group. A character $\chi \in \mathrm{Irr}(G)$ is called π-*special* if:

- both $\chi(1)$ and $o(\chi)$ are π-numbers,
- for any subnormal subgroup N of G and any irreducible constituent φ of χ_N, $o(\varphi)$ is a π-number.

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Theorem (Gajendragadkar, 1979)

Let G be a π-separable group. If α is a π-special character of G and β is a π'-special character of G, then $\alpha\beta$ is an irreducible character of G and this factorization is unique.

Definition

If a character can be written as the product of a π-special and a π'-special character, we say that it is π-*factorable*.

B_π-characters

Isaacs algorithm, 1982

Let G be a π-separable group and let $\chi \in \mathrm{Irr}(G)$. Through the algorithm described by Isaacs in the article *Characters of π-Separable Groups*, one can associate to χ an unique (up to conjugation) pair (W, μ) such that:

- $W \leq G$ and $\mu \in \mathrm{Irr}(W)$,
- $\mu^G = \chi$, and
- μ is π-factorable.

The pair (W, μ) is called a *nucleus* for χ.

Note that, as a consequence, every primitive character is π-factorable.

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Definition

Let $\chi \in \mathrm{Irr}(G)$ and let (W, μ) a nucleus for χ. If μ is π-special, then χ is a B_π*-character*.

Character restriction and B_π-characters

Why should we care about these characters?

Theorem (*Isaacs, 1974*)

If G is p-solvable, there exists a canonically defined set of characters $\mathrm{B}_{p'}(G)$ such that the restriction to p-regular elements realizes a bijection between $\mathrm{B}_{p'}(G)$ and $\mathrm{IBr}_p(G)$.

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Let χ^* be the restriction of a character χ to π-elements, i.e. to elements such that their order is a π-number. For any $\chi \in Char(G)$, we say that χ^* is a π-*partial character*. A π-partial character χ^* is in $I_\pi(G)$ if it is *irreducible*, i.e., if it cannot be written as a sum of two other π-partial characters.

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Theorem (*Isaacs, 1982*)

Let G be π-separable.

- *The restriction to π-elements realizes a bijection $B_\pi(G) \mapsto I_\pi(G)$.*
- *$|B_\pi(G)|$ is equal to the number of conjugacy classes of π-elements.*
- *The set $I_\pi(G)$ of irreducible π-partial characters is a basis for the class functions on π-elements.*

Fong characters

Let H be a Hall π-subgroup of G.

Theorem (Isaacs, 1982)

Let $\chi \in \mathrm{B}_\pi(G)$, then

a) *if $\alpha \in \mathrm{Irr}(H)$, then $\alpha(1) \geq [\chi_H, \alpha]\chi(1)_\pi$;*
b) *χ_H has an irreducible constituent α such that $\alpha(1) = \chi(1)_\pi$;*
c) *if α is as in b), then it is not an irreducible constituent of any other B_π-character different from χ.*

Characters as in b) are called *Fong characters* associated with χ.

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Theorem (Isaacs, 1984)

Let H be a Hall π-subgroup of a π-separable group G and let $\varphi \in \mathrm{Irr}(H)$. If φ is primitive, then it is a Fong character associated with some character $\chi \in \mathrm{B}_\pi(G)$. Moreover, $\eta \in \mathrm{Irr}(H)$ is a Fong character associated with χ if and only if $\eta = \varphi^g$ for some $g \in \mathrm{N}_G(H)$.

Bounding the p-length

p-length and character degrees

If G is a p-solvable group, the p-length $\ell_p(G)$ of G is the minimal number of p-quotients in a normal p-series of G.
For $n \in \mathbb{N}$, we write $\mathbb{Q}_n$ to denote the n-cyclotomic extension of $\mathbb{Q}$, i.e., an extension of $\mathbb{Q}$ by a primitive n-root of unity.

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Let G be p-solvable,

- if $p = 2$, $\ell_2(G) \leq |\mathrm{Irr}_{2',\mathbb{Q}}(G)|$ (Navarro, Tiep, 2007);
- for any p, $\ell_p(G) \leq \log_2(\left|\mathrm{Irr}_{p',\mathbb{Q}_p}(G)\right|)$ (Tent, 2013).

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Let $\mathrm{cd}_{p'}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_{p'}(G)\}$.

Theorem (Giannelli, Rizo, Schaeffer Fry, 2019)

Let G be finite and p odd. If $|\mathrm{cd}_{p'}(G)| = 2$, then G is solvable and $\ell_p(G) \leq 2$.

It is not difficult to prove that, if G is p-solvable, then $\ell_p(G) \leq |\mathrm{cd}_{p'}(G)|$.

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Question

Let G be p-solvable. Is it true that $\ell_p(G) \leq \left|\mathrm{cd}_{p',\mathbb{Q}_p}(G)\right|$?

Automorphisms of the field of values

Let G be p-solvable, P a Sylow p-subgroup of G, let $\chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ and let $\lambda \in \mathrm{Lin}(P)$ be a Fong character associated with χ.

Proposition

Let $\sigma \in \mathrm{Aut}(\mathbb{Q}_{|G|}/\mathbb{Q})$; if σ fixes λ, then it fixes χ.

If $\lambda^\sigma = \lambda$, then it is a Fong character associated with $\chi^\sigma \in \mathrm{B}_p(G)$ and, thus, $\chi^\sigma = \chi$.

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- If $\chi^\sigma = \chi$, then σ permutes the Fong characters of χ and $\chi_P = \sum_i \sum_{\lambda \in C_i} \lambda + \Delta$, with C_i orbits of σ and $p \mid \Delta(1)$.

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- $\chi(1) = \sum_i |C_i| \, \lambda(1) + \Delta(1)$ and, if $|C_i| \neq 1$ for all i, then $p \mid \chi(1)$, a contradiction.

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- $\chi(1) = \sum_i |C_i| \, \lambda(1) + \Delta(1)$ and, if $|C_i| \neq 1$ for all i, then $p \mid \chi(1)$, a contradiction.
- Since all the (primitive) Fong characters of χ are conjugated, one is fixed by σ iff all of them are.

Bound to the p-length

A B_p-character has values in $\mathbb{Q}_{|G|_p}$. Thus, a B_p-character has values in $\mathbb{Q}_p$ if and only if it is fixed by every $\sigma \in \mathrm{Aut}(\mathbb{Q}_{|G|_p}/\mathbb{Q}_p)$, which order is a p-power.

Corollary

Let $\chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ and let $\lambda \in \mathrm{Lin}(G)$ be a Fong character associated with χ. Then, χ has values in $\mathbb{Q}_p$ if and only if $o(\lambda) = p$.

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Thus, we have a way to control both existence, degree and field of values of the B_p-characters.

Theorem

Let G be p-solvable and $\mathrm{cd}^{\mathrm{B}_p}_{p',\mathbb{Q}_p}(G) = \{\chi(1) \mid \chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{p',\mathbb{Q}_p}(G)\}$, then $\ell_p(G) \leq \left|\mathrm{cd}^{\mathrm{B}_p}_{p',\mathbb{Q}_p}(G)\right|$.

Corollary

If G is a p-solvable group, $\ell_p(G) \leq \left|\mathrm{cd}_{p',\mathbb{Q}_p}(G)\right|$.

Any questions?

Thank you!