Toric Degenerations of Grassmannians and Schubert varieties

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Overview

1. Toric degeneration, Grassmannians and Schubert Varieties
2. Gröbner Degenerations and the Tropical Grassmannian
3. A Summary of Our Results

Toric Degenerations

A *toric degeneration* of a variety X is a flat family whose special fiber is a toric variety. All other fibers are isomorphic to X.

- Toric varieties are particularly well studied. Their algebraic invariants can often be given in terms of their polytope and fan.
- Let X be a variety and suppose we have a toric degeneration. We can read algebraic invariants of X from any fiber in particular the toric fiber.

Questions

- What are the toric degenerations of a given variety X?
- What structures exist to parametrise toric degenerations?

Grassmannians

Definition

The Grassmannian $Gr(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$.

Other ways to view $\mathrm{Gr}(k, n)$:

- The orbits of $k \times n$ matrices over $\mathbb{C}$ under the action of $\mathrm{GL}_k(\mathbb{C})$ on the left.
- The vanishing set of the Plücker ideal $I_{k,n}$ in $\mathbb{P}^{\binom{n}{k}-1}$.

Plücker Ideal

The Grassmannian is the vanishing set of the Plücker ideal $I_{k,n}$ in $\mathbb{P}^{\binom{n}{k}-1}$.

- $R = \mathbb{C}[P_I : I \subseteq [n], |I| = k]$, $S = \mathbb{C}[X]$ where $X = (x_{i,j})$ is a $k \times n$ matrix of variables
- $\phi : R \to S : P_I \mapsto \det(X_I)$, where X_I is the submatrix with columns I
- $I_{k,n} = \ker(\phi)$ the Plücker ideal generated by certain homogeneous quadrics

Example: $\mathrm{Gr}(2,4)$

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

$$\phi(P_{12}) = x_1 y_2 - x_2 y_1$$

$$\ker(\phi) = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle$$

A toric degeneration of $\mathrm{Gr}(2,4)$ is $\mathcal{F}_t$:

$$\mathcal{F}_t = \langle tP_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle,$$

$$\mathcal{F}_0 = \langle P_{13}P_{24} - P_{14}P_{23} \rangle.$$

Schubert Varieties

Definition

Let $w \in S_n$ be a permutation. The Schubert variety $X(w)$ has defining ideal $I_{k,n,w}$ which is the Plücker ideal $I_{k,n}$ where the variables $\{P_I : I \not\leq w\}$ are set to zero.

By '$I \leq w$' we mean: I is component-wise smaller than the set $\{w(1), \dots, w(k)\}$ after putting both sets in increasing order.

Example: Schubert varieties in $\mathrm{Gr}(2, 4)$

$$I_{2,4} = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle$$

Let $w = (1, 4, 2, 3)$ then the Plücker variables which are set to zero are

$$P_{24}, \; P_{34}.$$

$$I_{2,4,(1423)} = \langle P_{14}P_{23} \rangle.$$

Gröbner Degeneration

Our approach to finding toric generations comes from studying initial ideals.

Definition

Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal. Then each vector $w \in \mathbb{R}^n$ gives rise to a flat family whose special fiber is:

$$\mathrm{in}_w(I) = \{\mathrm{in}(f) : f \in I\}.$$

Where $\mathrm{in}(f)$ are all terms of f with lowest weight.

Example

- $R = \mathbb{C}[P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}]$
- $I = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle \subset R$
- $w = (1, 0, 0, 0, 0, 1) \in \mathbb{R}^6$

The initial ideal $\mathrm{in}_w(I)$ is a toric ideal:

$$\mathrm{in}_w(I) = \langle P_{13}P_{24} - P_{14}P_{23} \rangle$$

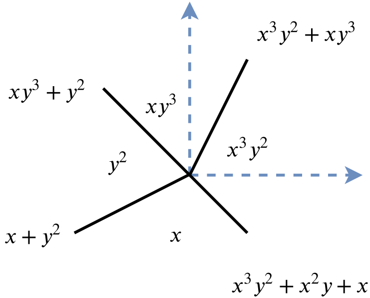
Gröbner Fan

The *Gröbner Fan of an ideal* $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is a fan in $\mathbb{R}^n$ which has one cone for each initial ideal I.

Example. Consider $I = \langle f \rangle$ where f is the polynomial:

$$f = x^3y^2 + x^2y + xy^3 + x + y^2.$$

Its Gröbner Fan is the fan in $\mathbb{R}^2$ whose cones are labelled by initial terms:



Tropicalisation

- A *generic* weight vector $w \in \mathbb{R}^n$ gives rise to a monomial ideal $\text{in}_w(I)$.
- Each $w \in \text{Trop}(I) \subset \mathbb{R}^n$ is a weight such that initial ideal $\text{in}_w(I)$ contains no monomials.

Question

Which weights $w \in \text{Trop}(I)$ give rise to toric initial ideals? i.e. $\text{in}_w(I)$ is a prime binomial ideal.

A few results

The Gelfan-Zeitlin degeneration gives one weight vector for each $\text{Gr}(k, n)$. For small values of k and n there are specific results:

- $\text{Gr}(2, n)$, all binomial initial ideals are prime. $\text{Trop}(\text{Gr}(2, n))$ can be seen as the space of phylogenetic trees (Speyer-Sturmfels 2003).
- $\text{Gr}(3, n)$, use *matching fields* to give families of toric degenerations (Mohammadi-Shaw 2018).

Our results

We generalise the family of toric degenerations described by so called *block diagonal matching fields* from $\mathrm{Gr}(3, n)$ (Mohammadi-Shaw 2018) to all Grassmannians.

Theorem

Each block diagonal matching field produces a toric degeneration of $\mathrm{Gr}(k, n)$. Equivalently, the Plücker forms are a SAGBI basis with respect to the weight vectors arising from block diagonal matching fields

- A toric degeneration of $\mathrm{Gr}(k, n)$ induces a flat family for each Schubert variety $X(w)$.
- The SAGBI basis (Subalgebra Analogue of Gröbner Basis for Ideals) allows us to study the ideals of Schubert varieties.
- We give a complete classification of block diagonal matching fields and permutations $w \in S_n$ which give rise to toric degenerations of $X(w)$.

References

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