Finiteness Properties for Totally Disconnected Locally Compact Groups

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Definition

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A locally compact group G is **TOTALLY DISCONNECTED** if the identity 1_G is its own connected component.

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Theorem (van Dantzig, 1932)

A topological group G is a TDLC-group if, and only if, G has a neighbourhood basis at 1_G *consisting of compact open subgroups.*

Motivation

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$G \;\; =$ locally compact group,

$G^o =$ connected component which contains the identity 1_G.

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G = locally compact group,

G^o = connected component which contains the identity 1_G.

G^o is a closed normal subgroup of G and so

$$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow G/G^\circ \longrightarrow 1$$

G° = **CONNECTED LC-GROUP** *and*

G/G° = **TDLC-GROUP.**

Examples

1. Discrete groups.
   E.g., every abstract group endowed with the discrete topology.
2. Profinite groups.
   E.g., the p-adic integers $\mathbb{Z}_p$.
3. Algebraic groups over non-archimedian local fields.
   E.g., $SL_2(\mathbb{Q}_p)$.
4. Graph automorphism groups.
   E.g., Group of automorphisms of a regular tree and Neretin's groups.

Modern Perspective

TDLC-groups are SIMULTANEOUSLY

geometric and topological groups.

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TDLC-groups are SIMULTANEOUSLY geometric and topological groups.

Profinite groups are trivial as geometric groups and Discrete groups are trivial as topological groups.

Geometric properties of (topological) groups

Definition

Let (X, d) be a metric space. The **large-scale geometric properties** are the properties that are invariant under quasi-isometry.

Hint: Regard the (topological) group as a metric space.

1. Number of ends,
2. Hyperbolicity,
3. Growth rate,
4. Amenability.

Cohomology of TDLC-groups

Rational Discrete Cohomology for TDLC-groups

(2016) Castellano, I. and Th. Weigel.
"Rational discrete cohomology for totally disconnected locally compact groups." Journal of Algebra.

Some Results:

1. (I. Castellano, 2018) Cohomological interpretation of Stallings' decomposition theorem for compactly generated TDLC-groups;

   * W. Dicks and M.J. Dunwoody, 1989.

2. (I. Castellano, 2018) Characterization of compactly presented TDLC-groups of rational discrete cohomological dimension ≤ 1;

   * M.J. Dunwoody, 1979.

3. (S. Arora, I. Castellano , E. Martinez-Pedroza, 2019) A subgroup theorem for hyperbolic TDLC-groups of cohomological dimension ≤ 2.

   * S. M. Gersten, 1996.

Finiteness properties for TDLC-groups

Classical finiteness properties

Discrete Groups	TDLC-groups
finite generation	**compact** generation
finite presentability	**compact** presentability

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Homological finiteness properties

Type FP_n over R	Type FP_n over $\mathbb{Q}$

(2016) Castellano, I. and Th. Weigel.

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Classical finiteness properties

Discrete Groups	TDLC-groups
finite generation	**compact** generation
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Homotopical finiteness properties

Type F_n

(2018) Castellano, I. and G. Corob Cook.
(2016) Sauer R. and W. Thumann.

Theorem (I. Castellano, G. Corob Cook, 2019)

Being of type FP_n *(resp. of type* F_n*) is a quasi-isometric invariant of compactly generated TDLC-groups.*

* The analogue for discrete groups is due to J. M. Alonso, 1993.

Thanks for your attention