MULTIPERMUTATION SOLUTIONS AND FACTORIZATIONS OF SKEW LEFT BRACES

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YANG-BAXTER AND ALGEBRAIC STRUCTURES

Definition

A set-theoretic solution to the Yang-Baxter equation is a tuple (X, r), where X is a set and $r : X \times X \longrightarrow X \times X$ a function such that (on X^3)

$$(\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r) = (r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X)\,.$$

For further reference, denote $r(x, y) = (\lambda_x(y), \rho_y(x))$.

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Definition

A set-theoretic solution (X, r) is called

- left (resp. right) non-degenerate, if λ_x (resp. ρ_y) is bijective,
- non-degenerate, if it is both left and right non-degenerate,
- involutive, if $r^2 = \mathrm{id}_{X\times X}$,

APPLICATIONS OF THE YANG-BAXTER EQUATION

- Statistical Physics (work of Yang and Baxter),
- Construction of Hopf Algebras,
- Knot theory (Reidemeister III, colourings),
- Quadratic algebras.

SOME EXAMPLES

Example

Let X be a set. Then, the twist $r(a, b) = (b, a)$ on $X \times X$ is an involutive non-degenerate solution. This is called the trivial solution.

Example (Lyubashenko)

Let X be a set. Let $f, g : X \longrightarrow X$ be maps. Then, $r(a, b) = (f(b), g(a))$ is a set-theoretic solution if $fg = gf$. If $g = f^{-1}$, then this set-theoretic solution is called a permutation solution.

SOLUTIONS LIKE LYUBASHENKO'S

Definition (Retraction)

Let (X, r) be an involutive non-degenerate set-theoretic solution. Define the relation $x \sim y$ on X, when $\lambda_x = \lambda_y$. Then, there exists a natural set-theoretic solution on $X/\sim$ called the retraction $\mathrm{Ret}(X, r)$.

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Denote for $n \geq 2$, $\mathrm{Ret}^n(X, r) = \mathrm{Ret}\left(\mathrm{Ret}^{n-1}(X, r)\right)$. If there exists a positive integer n such that $|\mathrm{Ret}^n(X, r)| = 1$, then (X, r) is called a multipermutation solution

THE STRUCTURE GROUP

Definition

Let (X, r) be a set-theoretic solution of the Yang-Baxter equation. Then the group

$$G(X, r) = \left\langle x \in X \mid xy = \lambda_x(y)\rho_y(x) \right\rangle,$$

is called the structure group of (X, r).

RECOVERING SOLUTIONS

Theorem (ESS, LYZ, S, GV)

Let (X, r) be a bijective non-degenerate solution to YBE, then there exists a unique solution r_G on the group $G(X, r)$ such that the associated solution r_G satisfies

$$r_G(i \times i) = (i \times i)r,$$

where $i : X \to G(X, r)$ is the canonical map.

WHY ARE MULTIPERMUTATION SOLUTIONS INTERESTING

Theorem (CJOBVAGI)

Let (X, r) be a finite involutive non-degenerate set-theoretic solution. The following statements are equivalent,

- *the solution (X, r) is a multipermutation solution,*
- *the group $G(X, r)$ is left orderable,*
- *the group $G(X, r)$ is diffuse,*
- *the group $G(X, r)$ is poly-$\mathbb{Z}$.*

CREATING SOLUTIONS ON $G(X, R)$ (1)

Definition (Rumo, CJO, GV)

Let $(B, +)$ and $(B, \circ)$ be groups on the same set B such that for any $a, b, c \in B$ it holds that

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

Then $(B, +, \circ)$ is called a skew left brace

If $(B, +)$ is abelian, one says that $(B, +, \circ)$ is a left brace.

CREATING SOLUTIONS ON $G(X,R)$ (1)

Definition (Rumo, CJO, GV)

Let $(B,+)$ and $(B,\circ)$ be groups on the same set B such that for any $a,b,c \in B$ it holds that

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

Then $(B,+,\circ)$ is called a skew left brace

If $(B,+)$ is abelian, one says that $(B,+,\circ)$ is a left brace. Denote for $a,b \in B$, the map $\lambda_a(b) = -a + a \circ b$. Then, $\lambda : (B,\circ) \longrightarrow \mathrm{Aut}(B,+) : a \mapsto \lambda_a$ is a well-defined group morphism.

CREATING SOLUTIONS ON $G(X, R)$ (2)

Theorem

Let $(B, +, \circ)$ be a skew left brace. Denote for any $a, b \in B$, the map $r_B(a, b) = (\lambda_a(b), \overline{(\overline{a} + b)} \circ b)$. Then (B, r_B) is a bijective non-degenerate solution. Moreover, if $(B, +)$ is abelian, then (B, r_B) is involutive.

Remark

Let (X, r) be a bijective non-degenerate set-theoretic solution. Then, $G(X, r)$ is a skew left brace.

STRUCTURE OF SKEW LEFT BRACES

Definition

Let $(B, +, \circ)$ be a skew left brace. Denote for any $a, b \in B$ the operation $a * b = \lambda_a(b) - b$ and denote for any positive integer $n > 1$, the set $B^{(n)} = B^{(n-1)} * B$. If there exists a positive integer n such that $B^{(n)} = 1$, we say that B is right nilpotent. If $B^{(2)} = 1$, we say that B is trivial.

Theorem (GIC)

Let (X, r) be an involutive non-degenerate set-theoretic solution. If the natural left brace $G(X, r)$ is right nilpotent, then the solutions $(G(X, r), r_G)$ and (X, r) are multipermutation solutions.

LEFT IDEALS AND IDEALS

Definition

Let $(B, +, \circ)$ be a skew left brace. Then, a (normal) subgroup I of $(B, +)$ such that $B * I \subseteq I$ is called a (strong) left ideal. Furthermore, if I is in addition a normal subgroup of $(B, \circ)$ then I is called an ideal of B.

Definition

Let $(B, +, \circ)$ be a skew left brace. If there exist left ideals I, J of B such that $I + J = B = J + I$, then B is called factorizable by I and J.

INTUITION: FACTORIZATIONS IN GROUPS

Theorem (Ito's Theorem)

Let $G = A + B$ be a factorized group. If A and B are both abelian, then G is metabelian (i.e. there exists an abelian normal subgroup N of G such that G/N is abelian).

Theorem

Let $G = A + B$ be a factorized group, where A and B are abelian. Then there exists a normal subgroup N of G contained in A or B.

Theorem (Kegel-Wielandt)

Let $G = A + B$ be a factorized group, where A and B are nilpotent. Then, G is solvable.

SURPRISING RESULTS

Theorem

Let $B = I + J$ be a factorized skew left brace. If I is a strong left ideal and both I and J are trivial skew left braces, then B is right nilpotent of class at most 4. If both are strong left ideals, then B is right nilpotent of class at most 3.

Theorem

Let $B = I + J$ be a factorized skew left brace. If I is a strong left ideal and both I and J are trivial skew left braces, then there exists an ideal N of B contained in I or J.

EXTENDING IS NOT POSSIBLE

Example (No Kegel-Wielandt)

There exists a simple (no non-trivial ideals) left brace of size 72, which is hence not solvable. By standard techniques one sees that this is factorizable by the additive Sylow subgroups.

Example (No relaxing conditions)

There exists a skew left brace of size 18 that is factorizable by 2 left ideals, both not strong left ideals. However, there is no ideal of the skew left brace contained in either of the left ideals.

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