Cohomology of finite Chevalley groups

Jack Saunders

16th September, 2019

University of Birmingham

Notation

Notation

- Let G be a finite Chevalley group

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$)

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.
- Let k be a field of characteristic $r > 0$.

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.
- Let k be a field of characteristic $r > 0$.
- Let $B \leq G$ be a Borel subgroup

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.
- Let k be a field of characteristic $r > 0$.
- Let $B \leq G$ be a Borel subgroup (think upper triangular).

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.
- Let k be a field of characteristic $r > 0$.
- Let $B \leq G$ be a Borel subgroup (think upper triangular).
- Let $\mathcal{L} := \mathrm{Ind}_B^G k$.

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.
- Let k be a field of characteristic $r > 0$.
- Let $B \leq G$ be a Borel subgroup (think upper triangular).
- Let $\mathcal{L} := \mathrm{Ind}_B^G k$.
- If V is a kG-module, then $V^G := \{v \in V \mid gv = v \, \forall g \in G\}$.

Notation

- Let G be a finite Chevalley group (think $\mathrm{SL}_n(q)$) defined in characteristic p.
- Let k be a field of characteristic $r > 0$.
- Let $B \leq G$ be a Borel subgroup (think upper triangular).
- Let $\mathcal{L} \coloneqq \operatorname{Ind}_B^G k$.
- If V is a kG-module, then $V^G \coloneqq \{v \in V \mid gv = v \,\forall g \in G\}$. This may be viewed as $\operatorname{Hom}_G(k, V)$.

Preliminaries

Definition

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$ with module homomorphisms $\partial^i \colon C^i \to C^{i+1}$

Preliminaries

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$ with module homomorphisms $\partial^i \colon C^i \to C^{i+1}$ such that $\partial^{i+1}\partial^i = 0$ for all i.

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$ with module homomorphisms $\partial^i \colon C^i \to C^{i+1}$ such that $\partial^{i+1}\partial^i = 0$ for all i. Cochain complexes are often represented as long sequences

$$\cdots \xrightarrow{\partial^{i-2}} C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \xrightarrow{\partial^{i+1}} \cdots .$$

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$ with module homomorphisms $\partial^i \colon C^i \to C^{i+1}$ such that $\partial^{i+1}\partial^i = 0$ for all i. Cochain complexes are often represented as long sequences

$$\cdots \xrightarrow{\partial^{i-2}} C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \xrightarrow{\partial^{i+1}} \cdots .$$

The i^{th} *cohomology* of $\boldsymbol{C}$

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$ with module homomorphisms $\partial^i \colon C^i \to C^{i+1}$ such that $\partial^{i+1}\partial^i = 0$ for all i. Cochain complexes are often represented as long sequences

$$\cdots \xrightarrow{\partial^{i-2}} C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \xrightarrow{\partial^{i+1}} \cdots .$$

The i^{th} *cohomology* of $\boldsymbol{C}$ is $\mathrm{H}^i(\boldsymbol{C})$

Definition

A *cochain complex* $\boldsymbol{C}$ is a collection of modules C^i for $i \in \mathbb{Z}$ with module homomorphisms $\partial^i \colon C^i \to C^{i+1}$ such that $\partial^{i+1}\partial^i = 0$ for all i. Cochain complexes are often represented as long sequences

$$\cdots \xrightarrow{\partial^{i-2}} C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \xrightarrow{\partial^{i+1}} \cdots.$$

The i^{th} *cohomology* of $\boldsymbol{C}$ is $\mathrm{H}^i(\boldsymbol{C}) \coloneqq \ker \partial^i / \operatorname{Im} \partial^{i-1}$.

Preliminaries

Preliminaries

Definition

Preliminaries

Definition

A kG-module P

Definition

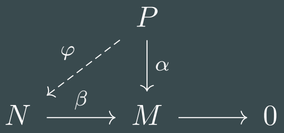
A kG-module P is said to be *projective*

Definition

A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes

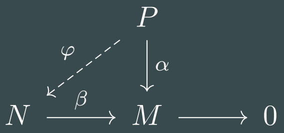
Definition

A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes



Definition

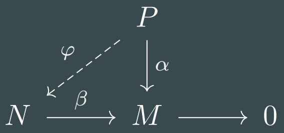
A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes



Definition

Definition

A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes



Definition

Given a kG-module V,

Preliminaries

Definition

A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes

$$\begin{array}{ccccc} & & P & & \\ & \overset{\varphi}{\swarrow} & \downarrow{\scriptstyle \alpha} & & \\ N & \xrightarrow{\beta} & M & \longrightarrow & 0 \end{array}$$

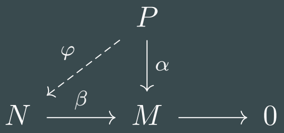
Definition

Given a kG-module V, a *projective resolution* $\boldsymbol{P}$ of V

Preliminaries

Definition

A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes



Definition

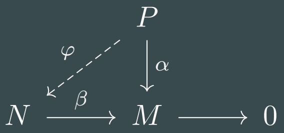
Given a kG-module V, a *projective resolution* $\boldsymbol{P}$ of V is a long exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0$$

Preliminaries

Definition

A kG-module P is said to be *projective* if there exists φ such that the following diagram commutes



Definition

Given a kG-module V, a *projective resolution* $\boldsymbol{P}$ of V is a long exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0$$

where each P_i is projective.

What even is cohomology?

What even is cohomology?

Definition

What even is cohomology?

Definition

Let V and W be kG-modules,

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

Now apply $\mathrm{Hom}_G(-, W)$ to this projective resolution to obtain the long exact sequence $\mathrm{Hom}_G(\boldsymbol{P}, W)$

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

Now apply $\mathrm{Hom}_G(-, W)$ to this projective resolution to obtain the long exact sequence $\mathrm{Hom}_G(\boldsymbol{P}, W)$:

$$0 \to \mathrm{Hom}_G(P_0, W) \to \mathrm{Hom}_G(P_1, W) \to \mathrm{Hom}_G(P_2, W) \to \cdots .$$

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

Now apply $\mathrm{Hom}_G(-, W)$ to this projective resolution to obtain the long exact sequence $\mathrm{Hom}_G(\boldsymbol{P}, W)$:

$$0 \to \mathrm{Hom}_G(P_0, W) \to \mathrm{Hom}_G(P_1, W) \to \mathrm{Hom}_G(P_2, W) \to \cdots .$$

We define $\mathrm{Ext}^i_G(V, W) \coloneqq \mathrm{H}^i(\mathrm{Hom}_G(\boldsymbol{P}, W))$

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

Now apply $\mathrm{Hom}_G(-, W)$ to this projective resolution to obtain the long exact sequence $\mathrm{Hom}_G(\boldsymbol{P}, W)$:

$$0 \to \mathrm{Hom}_G(P_0, W) \to \mathrm{Hom}_G(P_1, W) \to \mathrm{Hom}_G(P_2, W) \to \cdots .$$

We define $\mathrm{Ext}^i_G(V, W) \coloneqq \mathrm{H}^i(\mathrm{Hom}_G(\boldsymbol{P}, W))$, and define the i^{th} *group cohomology*

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

Now apply $\mathrm{Hom}_G(-, W)$ to this projective resolution to obtain the long exact sequence $\mathrm{Hom}_G(\boldsymbol{P}, W)$:

$$0 \to \mathrm{Hom}_G(P_0, W) \to \mathrm{Hom}_G(P_1, W) \to \mathrm{Hom}_G(P_2, W) \to \cdots .$$

We define $\mathrm{Ext}^i_G(V, W) := \mathrm{H}^i(\mathrm{Hom}_G(\boldsymbol{P}, W))$, and define the i^{th} *group cohomology* $\mathrm{H}^i(G, V) := \mathrm{Ext}^i_G(k, V)$.

What even is cohomology?

Definition

Let V and W be kG-modules, and take a projective resolution $\boldsymbol{P}$ of V.

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0.$$

Now apply $\mathrm{Hom}_G(-, W)$ to this projective resolution to obtain the long exact sequence $\mathrm{Hom}_G(\boldsymbol{P}, W)$:

$$0 \to \mathrm{Hom}_G(P_0, W) \to \mathrm{Hom}_G(P_1, W) \to \mathrm{Hom}_G(P_2, W) \to \cdots .$$

We define $\mathrm{Ext}^i_G(V, W) := \mathrm{H}^i(\mathrm{Hom}_G(\boldsymbol{P}, W))$, and define the i^{th} *group cohomology* $\mathrm{H}^i(G, V) := \mathrm{Ext}^i_G(k, V)$. Note that $\mathrm{Ext}^0_G(V, W) \cong \mathrm{Hom}_G(V, W)$ and so $\mathrm{H}^0(G, V) = V^G$.

Why should I care?

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M, N)$ and i-extensions of M by N.

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M, N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G, V)$ and conjugacy classes of complements to V in $V \rtimes G$.

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M, N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G, V)$ and conjugacy classes of complements to V in $V \rtimes G$.

iii) There is a 1–1 correspondence between $\mathrm{H}^2(G, V)$ and extensions of G by V with the given action of G on V.

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M, N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G, V)$ and conjugacy classes of complements to V in $V \rtimes G$.

iii) There is a 1–1 correspondence between $\mathrm{H}^2(G, V)$ and extensions of G by V with the given action of G on V.

iv) The *Schur multiplier*

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M,N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G,V)$ and conjugacy classes of complements to V in $V \rtimes G$.

iii) There is a 1–1 correspondence between $\mathrm{H}^2(G,V)$ and extensions of G by V with the given action of G on V.

iv) The *Schur multiplier* $M(G) = \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})$

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M,N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G,V)$ and conjugacy classes of complements to V in $V \rtimes G$.

iii) There is a 1–1 correspondence between $\mathrm{H}^2(G,V)$ and extensions of G by V with the given action of G on V.

iv) The *Schur multiplier* $M(G) = \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})$ classifies central extensions of perfect groups G.

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M,N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G,V)$ and conjugacy classes of complements to V in $V \rtimes G$.

iii) There is a 1–1 correspondence between $\mathrm{H}^2(G,V)$ and extensions of G by V with the given action of G on V.

iv) The *Schur multiplier* $M(G) = \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})$ classifies central extensions of perfect groups G. (quasisimple groups)

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M, N)$ and i-extensions of M by N.
ii) There is a 1–1 correspondence between $\mathrm{H}^1(G, V)$ and conjugacy classes of complements to V in $V \rtimes G$.
iii) There is a 1–1 correspondence between $\mathrm{H}^2(G, V)$ and extensions of G by V with the given action of G on V.
iv) The *Schur multiplier* $M(G) = \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ classifies central extensions of perfect groups G. (quasisimple groups)
v) Can use H^1 to determine the size of a minimal generating set for groups $V \rtimes G$.

Why should I care?

i) There is a 1–1 correspondence between $\mathrm{Ext}^i_R(M,N)$ and i-extensions of M by N.

ii) There is a 1–1 correspondence between $\mathrm{H}^1(G,V)$ and conjugacy classes of complements to V in $V \rtimes G$.

iii) There is a 1–1 correspondence between $\mathrm{H}^2(G,V)$ and extensions of G by V with the given action of G on V.

iv) The *Schur multiplier* $M(G) = \mathrm{H}^2(G,\mathbb{Q}/\mathbb{Z})$ classifies central extensions of perfect groups G. (quasisimple groups)

v) Can use H^1 to determine the size of a minimal generating set for groups $V \rtimes G$.

vi) and more...

Some useful tools

Theorem (Shapiro's Lemma)

Theorem (Shapiro's Lemma)

Let $H \leq G$,

Some useful tools

Theorem (Shapiro's Lemma)

Let $H \leq G$, V be a kH-module and W a kG-module.

Theorem (Shapiro's Lemma)

Let $H \leq G$, V be a kH-module and W a kG-module. Then

$$\operatorname{Ext}^i_G(\operatorname{Ind}^G_H V, W)$$

Theorem (Shapiro's Lemma)

Let $H \leq G$, V be a kH-module and W a kG-module. Then

$$\mathrm{Ext}^i_G(\mathrm{Ind}^G_H V, W) \cong \mathrm{Ext}^i_B(V, \mathrm{Res}^G_H W).$$

Some useful tools

Theorem (Shapiro's Lemma)

Let $H \leq G$, V be a kH-module and W a kG-module. Then

$$\mathrm{Ext}^i_G(\mathrm{Ind}^G_H V, W) \cong \mathrm{Ext}^i_B(V, \mathrm{Res}^G_H W).$$

When $i = 0$ this gives

$$\mathrm{Hom}_G(\mathrm{Ind}^G_H V, W) \cong \mathrm{Hom}_B(V, \mathrm{Res}^G_H W).$$

Some useful tools

Some useful tools

Theorem (Hochschild–Serre spectral sequence)

Some useful tools

Theorem (Hochschild–Serre spectral sequence)

Let $N \trianglelefteq G$ and let V be a kG-module.

Theorem (Hochschild–Serre spectral sequence)

Let $N \trianglelefteq G$ and let V be a kG-module. Then $\mathrm{H}^n(G, V)$ is a subquotient of

Some useful tools

Theorem (Hochschild–Serre spectral sequence)

Let $N \trianglelefteq G$ and let V be a kG-module. Then $\mathrm{H}^n(G,V)$ is a subquotient of

$$\bigoplus_{i+j=n} \mathrm{H}^i(G/N, \mathrm{H}^j(N,V)).$$

Some useful tools

Theorem (Hochschild–Serre spectral sequence)

Let $N \trianglelefteq G$ and let V be a kG-module. Then $\mathrm{H}^n(G, V)$ is a subquotient of

$$\bigoplus_{i+j=n} \mathrm{H}^i(G/N, \mathrm{H}^j(N, V)).$$

In particular, $\dim \mathrm{H}^n(G, V) \leq \sum_{i+j=n} \dim \mathrm{H}^i(G/N, \mathrm{H}^j(N, V))$.

Some useful tools

Theorem (Long exact sequence in group cohomology)

Theorem (Long exact sequence in group cohomology)

Suppose we have an exact sequence

$$0 \to V \to U \to W \to 0$$

of kG-modules.

Theorem (Long exact sequence in group cohomology)

Suppose we have an exact sequence

$$0 \to V \to U \to W \to 0$$

of kG-modules. Then we also have an exact sequence in cohomology

Theorem (Long exact sequence in group cohomology)

Suppose we have an exact sequence

$$0 \to V \to U \to W \to 0$$

of kG-modules. Then we also have an exact sequence in cohomology

$$0 \to V^G \to U^G \to W^G \to \cdots$$

Theorem (Long exact sequence in group cohomology)

Suppose we have an exact sequence

$$0 \to V \to U \to W \to 0$$

of kG-modules. Then we also have an exact sequence in cohomology

$$\cdots \to U^G \to W^G \to \mathrm{H}^1(G, V) \to \mathrm{H}^1(G, U) \to \mathrm{H}^1(G, W) \to \cdots$$

Some useful tools

Theorem (Long exact sequence in group cohomology)

Suppose we have an exact sequence

$$0 \to V \to U \to W \to 0$$

of kG-modules. Then we also have an exact sequence in cohomology

$$\cdots \to \mathrm{H}^1(G, W) \to \mathrm{H}^2(G, V) \to \mathrm{H}^2(G, U) \to \mathrm{H}^2(G, W) \to \cdots$$

What do we know?

What do we know?

Theorem (Parker–Stewart 2013)

Let G be a finite simple group of Lie type over k with Coxeter number h. Let V be an irreducible kG-module.

What do we know?

Theorem (Parker–Stewart 2013)

Let G be a finite simple group of Lie type over k with Coxeter number h. Let V be an irreducible kG-module. Then

$$\dim \mathrm{H}^1(G,V) \leq \max\left\{ \frac{z_p^{\lfloor h^3/6 \rfloor}}{z_p - 1}, \frac{1}{2}(h^2(3h-3)^3)^{\frac{h^2}{2}} \right\}$$

where $z_p = \lfloor h^3/6(1+\log_p(h-1)) \rfloor$.

What do we know?

What do we know?

Theorem (Guralnick–Tiep 2011)

What do we know?

Theorem (Guralnick–Tiep 2011)

Let G be a finite Chevalley group of (twisted) rank e with Weyl group W and let V be an irreducible kG-module, where $r \neq p$. Then

What do we know?

Theorem (Guralnick–Tiep 2011)

Let G be a finite Chevalley group of (twisted) rank e with Weyl group W and let V be an irreducible kG-module, where $r \neq p$. Then

i) If V is not a composition factor of $\mathcal{L}$, then $\mathrm{H}^1(G, V) = 0$.

What do we know?

Theorem (Guralnick–Tiep 2011)

Let G be a finite Chevalley group of (twisted) rank e with Weyl group W and let V be an irreducible kG-module, where $r \neq p$. Then

i) If V is not a composition factor of $\mathcal{L}$, then $\mathrm{H}^1(G, V) = 0$.
ii) If $V^B = 0$, then $\dim \mathrm{H}^1(G, V) \leq 1$ and there are at most four distinct such modules with $\mathrm{H}^1(G, V) \neq 0$.

What do we know?

Theorem (Guralnick–Tiep 2011)

Let G be a finite Chevalley group of (twisted) rank e with Weyl group W and let V be an irreducible kG-module, where $r \neq p$. Then

i) If V is not a composition factor of $\mathcal{L}$, then $\mathrm{H}^1(G, V) = 0$.
ii) If $V^B = 0$, then $\dim \mathrm{H}^1(G, V) \leq 1$ and there are at most four distinct such modules with $\mathrm{H}^1(G, V) \neq 0$.
iii) There are at most $|W|$ isomorphism classes of irreducible kG-modules V with $V^B \neq 0$

What do we know?

Theorem (Guralnick–Tiep 2011)

Let G be a finite Chevalley group of (twisted) rank e with Weyl group W and let V be an irreducible kG-module, where $r \neq p$. Then

i) If V is not a composition factor of $\mathcal{L}$, then $\mathrm{H}^1(G, V) = 0$.
ii) If $V^B = 0$, then $\dim \mathrm{H}^1(G, V) \leq 1$ and there are at most four distinct such modules with $\mathrm{H}^1(G, V) \neq 0$.
iii) There are at most $|W|$ isomorphism classes of irreducible kG-modules V with $V^B \neq 0$ and

$$\sum_{V \in \mathrm{Irr}_k G} \dim V^B \dim \mathrm{H}^1(G, V) \leq |W| + e.$$

What do we know?

What do we know?

Theorem (Guralnick–Kantor–Kassabov–Lubotzky 2007)

What do we know?

Theorem (Guralnick–Kantor–Kassabov–Lubotzky 2007)

If G is a finite quasisimple group, V a kG-module,

What do we know?

Theorem (Guralnick–Kantor–Kassabov–Lubotzky 2007)

If G is a finite quasisimple group, V a kG-module, then $\dim \mathrm{H}^2(G, V)$

What do we know?

Theorem (Guralnick–Kantor–Kassabov–Lubotzky 2007)

If G is a finite quasisimple group, V a kG-module, then $\dim \mathrm{H}^2(G, V) \leq 17.5 \dim V$.

What can we do?: An important step

What can we do?: An important step

Theorem (Guralnick–Tiep 2011)

What can we do?: An important step

Theorem (Guralnick–Tiep 2011)

If V is an irreducible kG-module

What can we do?: An important step

Theorem (Guralnick–Tiep 2011)

If V is an irreducible kG-module such that $V^B = 0$,

What can we do?: An important step

Theorem (Guralnick–Tiep 2011)

If V is an irreducible kG-module such that $V^B = 0$, we have that $\mathrm{H}^1(G, V)$ is the multiplicity of V^* in the socle of $\mathcal{L}/\mathcal{L}^G$.

What can we do?: A generalisation

What can we do?: A generalisation

Theorem (S., 2018)

What can we do?: A generalisation

Theorem (S., 2018)

Let V be a kG-module with $V^B = 0$, then

What can we do?: A generalisation

Theorem (S., 2018)

Let V be a kG-module with $V^B = 0$, then

$$\mathrm{H}^n(G, V) \cong \mathrm{Ext}_G^{n-1}(V^*, \mathcal{L}/\mathcal{L}^G) \cong \mathrm{H}^{n-1}(G, V \otimes \mathcal{L}/\mathcal{L}^G).$$

What can we do?: A generalisation

Theorem (S., 2018)

Let V be a kG-module with $V^B = 0$, then

$$\mathrm{H}^n(G, V) \cong \mathrm{Ext}_G^{n-1}(V^*, \mathcal{L}/\mathcal{L}^G) \cong \mathrm{H}^{n-1}(G, V \otimes \mathcal{L}/\mathcal{L}^G).$$

In particular, when $n = 1$ we get

$$\mathrm{H}^1(G, V) \cong \mathrm{Ext}_G^0(V^*, \mathcal{L}/\mathcal{L}^G).$$

What can we do?: A generalisation

Theorem (S., 2018)

Let V be a kG-module with $V^B = 0$, then

$$\mathrm{H}^n(G,V) \cong \mathrm{Ext}_G^{n-1}(V^*, \mathcal{L}/\mathcal{L}^G) \cong \mathrm{H}^{n-1}(G, V \otimes \mathcal{L}/\mathcal{L}^G).$$

In particular, when $n = 1$ we get

$$\mathrm{H}^1(G,V) \cong \mathrm{Hom}_G(V^*, \mathcal{L}/\mathcal{L}^G).$$

The proof

The proof

Lemma (Guralnick–Tiep 2011)

The proof

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$

The proof

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$ and V a kG-module.

The proof

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$ and V a kG-module. Then the following are equivalent:

The proof

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$ and V a kG-module. Then the following are equivalent:

i) $V^B \neq 0$,

The proof

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$ and V a kG-module. Then the following are equivalent:

i) $V^B \neq 0$,

ii) $V^A \neq 0$,

The proof

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$ and V a kG-module. Then the following are equivalent:

i) $V^B \neq 0$,

ii) $V^A \neq 0$,

iii) B has trivial composition factors on V,

Lemma (Guralnick–Tiep 2011)

Let $A := O_{r'}(B)$ and V a kG-module. Then the following are equivalent:

i) $V^B \neq 0$,

ii) $V^A \neq 0$,

iii) B has trivial composition factors on V,

iv) $(V^*)^B \neq 0$.

The proof

The proof

$$0 \to \mathcal{L}^G \to \mathcal{L} \to \mathcal{L}/\mathcal{L}^G \to 0$$

The proof

$$0 \longrightarrow \mathcal{L}^G \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{L}^G \longrightarrow 0$$

$$\Big\downarrow \mathrm{Hom}_G(V^*, -)$$

The proof

$$0 \longrightarrow \mathcal{L}^G \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{L}^G \longrightarrow 0$$

$$\rightsquigarrow \ \mathrm{Hom}_G(V^*, -)$$

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}^G) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/\mathcal{L}^G) \longrightarrow$$

The proof

$$0 \longrightarrow \mathcal{L}^G \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{L}^G \longrightarrow$$

$$\Big\downarrow \mathrm{Hom}_G(V^*,-)$$

$$0 \to \mathrm{Hom}_G(V^*, \mathcal{L}^G) \to \mathrm{Hom}_G(V^*, \mathcal{L}) \to \mathrm{Hom}_G(V^*, \mathcal{L}/\mathcal{L}^G) \to \mathrm{Ex}$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}^G) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/\mathcal{L}^G) \longrightarrow \mathrm{Ex}$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}^G) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/\mathcal{L}^G) \longrightarrow \mathrm{Ex}$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, k) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G($$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, k) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G($$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, k) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G($$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(k, V) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V$$

The proof

$$0 \longrightarrow V^G \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow$$

The proof

$$0 \longrightarrow 0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{E}$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V$$

The proof

$$0 \to \mathrm{Hom}_G(V^*, \mathcal{L}) \to \mathrm{Hom}_G(V^*, \mathcal{L}/k) \to \mathrm{Ext}^1_G(V^*, k) \to \mathrm{Ext}^1_G(V$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(\mathcal{L}, V) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V^*$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(\mathrm{Ind}_B^G\, k, V) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}$$

The proof

$$0 \longrightarrow \mathrm{Hom}_B(k, V) \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V^*$$

The proof

$$0 \longrightarrow V^B \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow$$

The proof

$$0 \to 0 \to \mathrm{Hom}_G(V^*, \mathcal{L}/k) \to \mathrm{Ext}^1_G(V^*, k) \to \mathrm{Ext}^1_G(V^*, \mathcal{L}) \to \cdots$$

The proof

$$0 \longrightarrow \operatorname{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \operatorname{Ext}^1_G(V^*, k) \longrightarrow \operatorname{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(V^*, k) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^1_G(k, V) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_G(\mathcal{L}, V) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_G(\mathrm{Ind}_B^G\, k, V) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{Ext}^1_B(k, V) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{H}^1(B, V) \longrightarrow \cdots$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B)$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$.

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

What can we say about $\mathrm{H}^n(B,V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B,V) \leq \sum_{i=0}^{n} \dim \mathrm{H}^i(B/A, \mathrm{H}^{n-i}(A,V))$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B, V) \leq \sum_{i=0}^{n-1} \dim \mathrm{H}^i(B/A, H^{n-i}(A, V)) + \dim \mathrm{H}^n(B/A, V^A).$$

The proof

What can we say about $\mathrm{H}^n(B,V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B,V) \leq \sum_{i=0}^{n-1} \dim \mathrm{H}^i(B/A, H^{n-i}(A,V)) + \dim \mathrm{H}^n(B/A, V^A).$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B, V) \leq \sum_{i=0}^{n-1} \dim \mathrm{H}^i(B/A, 0) + \dim \mathrm{H}^n(B/A, V^A).$$

The proof

What can we say about $\mathrm{H}^n(B,V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B,V) \leq 0 + \dim \mathrm{H}^1(B/A, V^A).$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B, V) \leq \dim \mathrm{H}^1(B/A, V^A).$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B, V) \leq \dim \mathrm{H}^1(B/A, V^A).$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B, V) \leq 0.$$

The proof

What can we say about $\mathrm{H}^n(B, V)$ $(n \geq 1)$ when $V^B = 0$?

Let $A := O_{r'}(B) \trianglelefteq B$. We apply the Hochschild–Serre spectral sequence to get

$$\dim \mathrm{H}^n(B, V) = 0.$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow \mathrm{H}^1(B, V) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Hom}_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^1(G, V) \longrightarrow 0 \longrightarrow \cdots$$

The proof

$$0 \to \mathrm{Hom}_G(V^*, \mathcal{L}/k) \to \mathrm{H}^1(G, V) \to 0 \to \cdots$$

The proof

$$0 \to \mathrm{Hom}_G(V^*, \mathcal{L}/k) \xrightarrow{\sim} \mathrm{H}^1(G, V) \to 0 \to \cdots$$

The proof

$$0 \to \mathrm{Hom}_G(V^*, \mathcal{L}/k) \xrightarrow{\sim} \mathrm{H}^1(G, V) \to 0 \to \cdots$$

The proof

$$0 \longrightarrow \operatorname{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \operatorname{Ext}^2_G(V^*, k) \longrightarrow \operatorname{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^2_G(V^*, k) \longrightarrow \mathrm{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^2_G(V^*, k) \longrightarrow \mathrm{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{Ext}^2_G(V^*, k) \longrightarrow \mathrm{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \operatorname{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^2(G, V) \longrightarrow \operatorname{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^2(G, V) \longrightarrow \mathrm{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \longrightarrow \operatorname{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^2(G, V) \longrightarrow \operatorname{Ext}^2_G(V^*, \mathcal{L}) \longrightarrow \cdots$$

The proof

$$0 \to \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \to \mathrm{H}^2(G, V) \to \mathrm{Ext}^2_G(V^*, \mathcal{L}) \to \cdots$$

The proof

$$0 \longrightarrow \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \longrightarrow \mathrm{H}^2(G, V) \longrightarrow \mathrm{H}^2(B, V) \longrightarrow \cdots$$

The proof

$$0 \to \mathrm{Ext}^1_G(V^*, \mathcal{L}/k) \to \mathrm{H}^2(G, V) \to 0 \to \cdots$$

The proof

$$0 \to \operatorname{Ext}^1_G(V^*, \mathcal{L}/k) \xrightarrow{\sim} \mathrm{H}^2(G, V) \to 0 \to \cdots$$

The proof

$$0 \longrightarrow \operatorname{Ext}_G^{n-1}(V^*, \mathcal{L}/k) \xrightarrow{\sim} \mathrm{H}^n(G, V) \longrightarrow 0 \longrightarrow \cdots$$

Future/other work

Future/other work

- Determine cohomology of rank one groups.

Future/other work

- Determine cohomology of rank one groups. $\mathrm{PSL}_2(q)$ almost completed,

Future/other work

- Determine cohomology of rank one groups. $\mathrm{PSL}_2(q)$ almost completed, $\mathrm{PSU}_3(q)$ in progress, ${}^2\mathrm{G}_2(q)$ afterwards.

Future/other work

- Determine cohomology of rank one groups. $\mathrm{PSL}_2(q)$ almost completed, $\mathrm{PSU}_3(q)$ in progress, ${}^2\mathrm{G}_2(q)$ afterwards.
- General result, bound on cohomology?

Future/other work

- Determine cohomology of rank one groups. $\mathrm{PSL}_2(q)$ almost completed, $\mathrm{PSU}_3(q)$ in progress, ${}^2\mathrm{G}_2(q)$ afterwards.
- General result, bound on cohomology?
- Can we say anything useful for modules with $V^B \neq 0$?

Future/other work

- Determine cohomology of rank one groups. $\mathrm{PSL}_2(q)$ almost completed, $\mathrm{PSU}_3(q)$ in progress, ${}^2\mathrm{G}_2(q)$ afterwards.
- General result, bound on cohomology?
- Can we say anything useful for modules with $V^B \neq 0$? (sort of)

Future/other work

- Determine cohomology of rank one groups. $\mathrm{PSL}_2(q)$ almost completed, $\mathrm{PSU}_3(q)$ in progress, ${}^2\mathrm{G}_2(q)$ afterwards.
- General result, bound on cohomology?
- Can we say anything useful for modules with $V^B \neq 0$? (sort of)

Thanks for listening!