### Groups with some subgroups complemented

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This is a joint work<sup>1</sup> with **Sergio Camp-Mora**, from Universitat Politècnica de València.



S. Camp-Mora, C. Monetta,

Groups with some families of subgroups complemented, submitted.

<sup>1</sup>Funded by GNSAGA

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- $H \cap K = 1$

The subgroup K is called a complement of H in G.

#### Example

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**★** The alternating group of degree n is always complemented in the symmetric group of degree n.

# ★ If K is a complement of H in G, then K detects a complete set of representative of both left and right cosets of H in G.

★ In a finite group every normal Hall subgroup has a complement, where H is a normal Hall subgroup of a group G if (|H|, |G/H|) = 1.

★ The concept of *complement* generalizes that of direct and semidirect factor.

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### A group G is said to be a C-group if all its subgroups are complemented.

#### Remark

The class of all *C*-groups is closed with respect to forming subgroups, images and direct products.

Let H be a subgroup of a C-group G. Then, every subgroup L of H admits a complement K in G, that is

• 
$$G = LK$$

•  $L \cap K = 1$ 

Then  $K_1 = K \cap H$  is a complement of L in H because

• 
$$LK_1 = L(K \cap H) = LK \cap H = G \cap H = H$$

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Recall that a group G has the subgroup lattice complemented if for every  $H \le G$  there exists  $K \le G$  such that

- $G = \langle H, K \rangle$
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### Theorem (P. Hall)

### For a finite group G, the following condition are equivalent.

- G is a C-group.
- ② G is a supersoluble and the subgroup lattice of G is complemented.
- G is isomorphic to a subgroup of a direct product of groups of squarefree orders.



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 $1 \leq H_1 \leq H_2 \leq \cdots$ 

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### What about a group if only some of its subgroups are complemented?

Theorem (P. Hall)

A finite group G is soluble

### if and only if

every Sylow subgroup of G is complemented.



P. Hall, Complemented groups, J. of London Math. Soc., **12** (1937), 201-204.

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**Theorem (A. Ballester-Bolinches, Guo Xiuyun)**  *A finite group G has all <u>minimal subgroups</u> complemented <i>if and only if G is supersoluble with elementary abelian Sylow subgroups* 

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#### Corollary (A. Ballester-Bolinches, Guo Xiuyun)

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Let G be a **periodic** group.

Then G has every minimal subgroup complemented

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## **Cyclic subgroups**

Let G be a locally soluble group. Then the following properties are equivalent.

- **1** Every cyclic subgroup of G is complemented
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## A group whose cyclic subgroups are complemented, is necessarily a C-group?

In general, the answer is NO!

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#### In general, the answer is NO!

Let  $H = \text{Dr}_{i \ge 1} \langle x_i \rangle$  and  $B = \text{Dr}_{j \ge 1} \langle y_j \rangle$  where  $x_i$  has order 3 and  $y_j$  has order 2 for every  $i, j \ge 1$ .

We define the following action:

$$x_i^{y_n} = \begin{cases} x_{i+2^{n-1}} & \text{if } i \equiv 1, \dots, 2^{n-1} \mod 2^n \\ x_{i-2^{n-1}} & \text{if } i \equiv 2^{n-1} + 1, \dots, 2^n \mod 2^n \end{cases}$$

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#### A group G is said to have finite (Prüfer) rank r if

- every finitely generated subgroup of G can be generated by at most r elements
- r is the least positive integer with such property

If there is no such an r, the group G has infinite rank.

#### Example

★  $C_{p^{\infty}}$ ,  $\mathbb{Q}$  are locally cyclic groups, so they have rank 1

- **★** Every polycyclic group has finite rank
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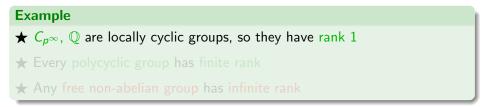
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#### Theorem (A. I. Mal'cev)

A locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank.



A. I. Mal'cev,

*On certain classes of infinite soluble groups*, Amer. Math. Soc. Translation **2** (1956), 1-21.

### Theorem (V. P. Šunkov)

A locally finite group with all abelian subgroups of finite rank itself has finite rank.



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#### Theorem (Y. I. Merzljakov)

There exist locally soluble groups of infinite rank in which every abelian subgroup has finite rank.

Y. I. Merzljakov,

Locally soluble groups of finite rank, Algebra and Logic 8 (1969), 686-690.

Theorem (M. R. Dixon, M. J. Evans, H. Smith)

Let c be a positive integer and let G be a locally soluble group of infinite rank whose proper subgroups of infinite rank are nilpotent with class at most c.

Then G is nilpotent of class at most c.

M. R. Dixon, M. J. Evans, H. Smith, Locally (soluble-by-finite) groups with all proper insoluble subgroups of finite rank, Arch. Math. (Basel) **68** (1997), 100-109.

#### Theorem (M. R. Dixon, M. J. Evans, H. Smith)

Let k be a positive integer and let G be a soluble group of infinite rank whose proper subgroups of infinite rank have derived length at most k. Then G has derived length at most k.

M. R. Dixon, M. J. Evans, H. Smith, Locally (soluble-by-finite) groups with all proper non-nilpotent subgroups of finite rank, J. Pure Appl. Algebra **135** (1999), 33-43. **Theorem (M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi)** Let G be a locally soluble group of infinite rank. If all proper subgroups of infinite rank of G have locally finite commutator subgroup, then G' is locally finite.

M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi, Groups with restrictions on subgroups of infinite rank, Rev. Mat. Iberoamericana 2 (2014).

Let G be a **periodic** locally soluble group of infinite rank.

If every infinite rank subgroup of G is complemented in G,

then G is a C-group.

#### **Open Problem**

Let G be a locally soluble group of infinite rank.

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## Thank you for the attention!