Groups with some subgroups complemented

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Young Researchers Algebra Conference 2019

17th September 2019

This is a joint work[1] with **Sergio Camp-Mora**, from Universitat Politècnica de València.

S. Camp-Mora, C. Monetta,
Groups with some families of subgroups complemented, submitted.

[1] Funded by GNSAGA

Complemented subgroups

A subgroup H of a group G is said to be complemented in G if there exist a subgroup K of G such that

- $G = HK$
- $H \cap K = 1$

The subgroup K is called a complement of H in G.

Example

★ The alternating group of degree n is always complemented in the symmetric group of degree n.

★ If $G = \langle x \rangle$ is the cyclic group of order 4, then the subgroup $H = \langle x^2 \rangle$ is not complemented in G.

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Some well-known facts

★ If K is a complement of H in G, then K detects a complete set of representative of both left and right cosets of H in G.

★ In a finite group every normal Hall subgroup has a complement, where H is a normal Hall subgroup of a group G if $(|H|, |G/H|) = 1$.

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C-groups

A group G is said to be a C-group if all its subgroups are complemented.

Remark

The class of all C-groups is closed with respect to forming subgroups, images and direct products.

Let H be a subgroup of a C-group G. Then, every subgroup L of H admits a complement K in G, that is

- $G = LK$
- $L \cap K = 1$

Then $K_1 = K \cap H$ is a complement of L in H because

- $LK_1 = L(K \cap H) = LK \cap H = G \cap H = H$
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A characterization of finite C-groups

Recall that a group G has the subgroup lattice complemented if for every $H \leq G$ there exists $K \leq G$ such that

- $G = \langle H, K \rangle$
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Theorem (P. Hall)

*For a **finite** group G, the following condition are equivalent.*

1. *G is a C-group.*
2. *G is a supersoluble and the subgroup lattice of G is complemented.*
3. *G is isomorphic to a subgroup of a direct product of groups of squarefree orders.*

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Recall that a group G is hypercyclic if there exists an ascending normal series

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such that H_i is a normal subgroup of G and H_{i+1}/H_i is cyclic for every $i \geq 1$.

Theorem (N. V. Cernikova)

For a group G, the following condition are equivalent.

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Question

What about a group if only some of its subgroups are complemented?

Sylow subgroups complemented

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A **finite** *group G is soluble*

if and only if

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Theorem (Z. Arad, M. B. Ward)

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A. Ballester-Bolinches and **Guo Xiuyun** analyzed the class of groups in which every minimal subgroup is complemented, where a minimal subgroup is a subgroup of prime order.

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that is, groups whose finitely generated subgroups are soluble.

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Let G be a locally soluble group. Then the following properties are equivalent.

1. *Every* **cyclic subgroup** *of G is complemented*
2. *G is periodic and every* **minimal subgroup** *of G is complemented*
3. *G is the semidirect product of $A = \mathrm{Dr}_{i \in I} A_i$ by $B = \mathrm{Dr}_{j \in J} B_j$ where all the A_i and B_j have prime order, and A* **has a set of maximal subgroups normal in** *G* **with trivial intersection**.

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Question

A group whose cyclic subgroups are complemented,

is necessarily a C-group?

In general, the answer is NO!

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Negative Answer

Counter-example (S. Camp-Mora, C. M.)

Let $H = \mathrm{Dr}_{i \geq 1} \langle x_i \rangle$ and $B = \mathrm{Dr}_{j \geq 1} \langle y_j \rangle$ where x_i has order 3 and y_j has order 2 for every $i, j \geq 1$.

We define the following action:

$$ {x_i}^{y_n} = \begin{cases} x_{i+2^{n-1}} & \text{if } \; i \equiv 1, \ldots, 2^{n-1} \mod 2^n \\ x_{i-2^{n-1}} & \text{if } \; i \equiv 2^{n-1}+1, \ldots, 2^n \mod 2^n \end{cases} $$

for every $i, n \geq 1$.

Then, $G = \langle H, B \rangle = HB$ has every minimal subgroup complemented,

but G does not contain any normal minimal subgroup,

that is, G is not a C-group.

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Families of subgroups

The second family of subgroups we considered is that of

Infinite rank subgroups

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Prüfer rank

A group G is said to have finite (Prüfer) rank r if

- every finitely generated subgroup of G can be generated by at most r elements
- r is the least positive integer with such property

If there is no such an r, the group G has infinite rank.

Example

★ C_{p^∞}, $\mathbb{Q}$ are locally cyclic groups, so they have rank 1

★ Every polycyclic group has finite rank

★ Any free non-abelian group has infinite rank

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Why locally soluble groups?

Theorem (A. I. Mal'cev)

A locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank.

A. I. Mal'cev,
On certain classes of infinite soluble groups, Amer. Math. Soc. Translation **2** (1956), 1-21.

Theorem (V. P. Šunkov)

A locally finite group with all abelian subgroups of finite rank itself has finite rank.

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There exist locally soluble groups of infinite rank in which every abelian subgroup has finite rank.

Y. I. Merzljakov,
Locally soluble groups of finite rank, Algebra and Logic **8** (1969), 686-690.

Strong influence

Theorem (M. R. Dixon, M. J. Evans, H. Smith)

Let c be a positive integer and let G be a locally soluble group of infinite rank whose proper subgroups of infinite rank are nilpotent with class at most c.

Then G is nilpotent of class at most c.

M. R. Dixon, M. J. Evans, H. Smith,
Locally (soluble-by-finite) groups with all proper insoluble subgroups of finite rank, Arch. Math. (Basel) **68** (1997), 100-109.

Strong influence

Theorem (M. R. Dixon, M. J. Evans, H. Smith)

Let k be a positive integer and let G be a soluble group of infinite rank whose proper subgroups of infinite rank have derived length at most k.

Then G has derived length at most k.

M. R. Dixon, M. J. Evans, H. Smith,
Locally (soluble-by-finite) groups with all proper non-nilpotent subgroups of finite rank, J. Pure Appl. Algebra **135** (1999), 33-43.

Strong influence

Theorem (M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi)

Let G be a locally soluble group of infinite rank. If all proper subgroups of infinite rank of G have locally finite commutator subgroup,

then G' is locally finite.

M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi,
Groups with restrictions on subgroups of infinite rank, Rev. Mat. Iberoamericana **2** (2014).

Infinite rank subgroups Complemented

Theorem (S. Camp-Mora, C. M.)

Let G be a **periodic** *locally soluble group of infinite rank.*

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Thank you for the attention!