On serial group rings of central extensions of simple groups

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Let R be an associative ring with unity.
A (left) R-module M is called **uniserial** if the lattice of its submodules is totally ordered by inclusion.

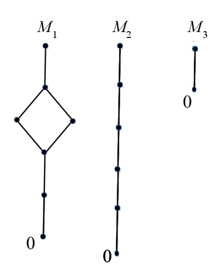


Figure 1: M_1 is not uniserial, M_2 is uniserial, M_3 is simple

A ring R is called **serial**, if both the left regular module ${}_RR$ and the right regular module R_R are a direct sum of uniserial modules:

$$ {}_RR = Re_1 \oplus \dots \oplus Re_n, \quad R_R = e_1R \oplus \dots \oplus e_nR $$

(where $e_i = e_i^2$ is a primitive idempotent of R)

Suppose G is a finite group,
F is a field of characteristic $p > 0$,
FG is the group ring (group algebra) of G over F.

Theorem 1 (H. Maschke)

FG is semisimple $\quad \Leftrightarrow \quad p \nmid |G|$.

Moreover, if $p \nmid |G|$, then

$$FG = \mathrm{M}_{n_1}(D_1) \oplus \ldots \oplus \mathrm{M}_{n_k}(D_k),$$

where D_i is a finite dimensional division algebra over F.

Question. What is the structure of FG when p divides $|G|$?

Problem. To describe all pairs (F, G), such that the group ring FG is serial.

Theorem 2 (D.G. Higman)

If FG is serial, then a p-Sylow subgroup P of G is cyclic.

Theorem 3 (I. Murase)

If F is a field of characteristic p and G is a p-nilpotent finite group with a cyclic p-Sylow subgroup, then the group ring FG is serial.

Theorem 4 (K. Morita)

If F is an algebraically closed field of characteristic p and G is a p-solvable finite group with a cyclic p-Sylow subgroup, then the group ring FG is serial.

Theorem 5 (D. Eisenbud and P. Griffith)

Let F' be a subfield of F. Then the ring $F'G$ iff FG is serial.

Theorem 6

If G is a p-nilpotent group with a cyclic p-Sylow subgroup, then FG is a principal ideal ring (and therefore it is serial).

In the decomposition $R_R = \oplus_{i=1}(e_i R)^{k_i}$, the number k_i is called the *multiplicity* of the projective module $P_i = e_i R$.

Theorem 7

Let G be a p-solvable group with a cyclic p-Sylow subgroup. Then the multiplicities of indecomposable projective modules in each block of FG coincide.

i.e. if $e_i R$ and $e_j R$ are in the same block of FG, then $k_i = k_j$.

Examples

Let F be a field of characteristic 3.

1. $FQ_8 = F \oplus F \oplus F \oplus F \oplus M_2(F)$.

2. $F\mathrm{SL}(2,3) = M_3(F) \oplus V \oplus M_2(V)$, where $V = F[x]/(x^3)$ is a chain ring of length 3. ($\mathrm{SL}(2,3)$ is a p-nilpotent group of order 24 with cyclic Sylow p-subgroup $P \cong C_3$).

3. Let $G = 2.S_4^- \cong SL(2,3).C_2$ (the double covering of S_4). Then G is 3-solvable group of order 48, and $P \cong C_3$.

Proposition 8

The group ring FG is serial. Furthermore,

1) If $F = \mathbb{F}_3$, then $FG = M_3(F) \oplus M_3(F) \oplus B \oplus M_2(W)$, where B is the serial block

$$\begin{pmatrix} F[x] & F[x] \\ xF[x] & F[x] \end{pmatrix} / \begin{pmatrix} x^2F[x] & xF[x] \\ x^2F[x] & x^2F[x] \end{pmatrix},$$

and $W = \mathbb{F}_9[y,\alpha]/(y^3)$ is the factor of the skew polynomial ring with the Frobenius automorphism $\lambda \mapsto \lambda^3$.

2) If $F = \mathbb{F}_9$, then $FG = M_3(F) \oplus M_3(F) \oplus B \oplus M_2(B)$.

$\mathrm{Irr}(G)$ is the set of irreducible ordinary characters of the group G;
$\mathrm{IBr}(G)$ is the set of irreducible p-modular (Brauer) characters of the group G;
Let $\chi \in \mathrm{Irr}(G)$, and let $\hat{\chi}$ be a restriction of χ on the set of p'-elements.

$$\hat{\chi} = \sum_{\phi \in \mathrm{IBr}(G)} d_{\chi\phi}\phi.$$

Brauer graph is a undirected graph, whose vertex set is $\mathrm{Irr}(G)$, and whose set of edges is $\mathrm{IBr}(G)$. Two vertices χ, ψ are linked by an edge, if $\exists \phi \in \mathrm{IBr}(G) :\ d_{\chi\phi} \neq 0,\ d_{\psi\phi} \neq 0$.

Exceptional vertex is a vertex which contains more then one character (from $\mathrm{Irr}(G)$).

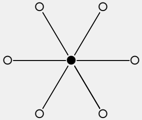
A connected component of the Brauer graph is called a *p-block* of G.
If F is an algebraically closed field of characteristic p, then
$\{p\text{-blocks of } G\} \longleftrightarrow \{\text{blocks of } FG\}$.

We call $\phi \in \mathrm{IBr}(G)$ *liftable* if there exist $\chi \in \mathrm{Irr}(G)$ such that $\hat{\chi} = \phi$.

Let F be an algebraically closed field of characteristic p.

Fact 9 (Janusz G.)

Let B be a p-block of G with (nontrivial) cyclic defect group. Then the following are equivalent:
a) every irreducible p-modular character of B is liftable;
b) the Brauer tree of B is a star with the exceptional vertex (if it exists) at the center;
c) B is serial.



Corollary 10

The group ring FG is serial if and only if the Brauer tree of any p-block of G is a star with the exceptional vertex at the center.

Example. $G = A_5$, $p = 5$

Let $G = A_5$ and $p = 5$.

	ϕ_1	ϕ_2
χ_1	1	0
χ_2	0	1
χ_3	0	1
χ_4	1	1

	ϕ_3
χ_5	1

$$\circ_{\chi_1} \overset{\phi_1}{\rule{3em}{0.4pt}} \circ_{\chi_4} \overset{\phi_2}{\rule{3em}{0.4pt}} \bullet_{\chi_2,\chi_3}$$

$$\circ_{\chi_5 = \phi_3}$$

- the Brauer tree of any p-block of A_5 is a star
  (but the exceptional vertex is not at the center);
- FA_5 is not serial if $\mathrm{char} F = 5$.

If $F = \mathbb{F}_5$, $R_R = P_1^5 \oplus P_2^3 \oplus P_3$, where

$$P_1 = \left(S_5 \right), \quad P_2 = \left(\begin{smallmatrix} & S_3 & \\ S_1 & & S_3 \\ & S_3 & \end{smallmatrix} \right) \quad \text{and} \quad P_3 = \left(\begin{smallmatrix} S_1 \\ S_3 \\ S_1 \end{smallmatrix} \right),$$

(S_i are simple modules).

Theorem 11

Let G be a finite simple group and let F be a field of characteristic p dividing the order of G. Then the group ring FG is serial if and only if one of the following holds.
1) $G = C_p$.
2) $G = \mathrm{PSL}_2(q)$, $q \neq 2$ or $G = \mathrm{PSL}_3(q)$, where $q \equiv 2, 5 \pmod 9$, and $p = 3$.
3) $G = \mathrm{PSL}_2(q)$ or $G = \mathrm{PSU}_3(q^2)$, where p divides $q-1$, and $p > 2$.
4) $G = Sz(q)$, $q = 2^{2n+1}$, $n \geq 1$, where either $p > 2$ divides $q-1$, or $p = 5$ divides $q + r + 1$, $r = 2^{n+1}$, but 25 does not divide this number.
5) $G = {}^2G_2(q^2)$, $q^2 = 3^{2n+1}$, $n \geq 1$, where either $p > 2$ divides $q^2 - 1$, or $p = 7$ divides $q^2 + \sqrt{3}q + 1$, but 49 does not divide this number.
6) $G = M_{11}$ and $p = 5$.
7) $G = J_1$ and $p = 3$.

Example. $G = A_5 \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$, $F = \mathbb{F}_3$.

Known fact. If FG is serial and $H \triangleleft G$, then $F(G/H)$ is serial.

Open question. If FG is serial and $H \triangleleft G$, then FH is serial?

Fact 12 (H.I. Blau, N. Naehrig)

Suppose G is a non-p-solvable group with a nontrivial cyclic p-Sylow subgroup P, and F is a field of characteristic p.
Then G has a unique minimal normal subgroup K, such that $K \supsetneq O_{p'}$. Moreover, $K \supset P$, and $H := K/O_{p'}$ is a simple non-abelian group.

Moreover, there is a normal series

$$1 \subseteq O_{p'}(G) \subseteq K \subseteq G.$$

Conjecture 13

FG is serial $\iff$ FH is serial.

It is true if $|G| \leq 10^4$.

Group rings of Suzuki groups

Let $p = 7$.

Let $H = Sz(8)$, one of the Suzuki groups $Sz(2^{2n+1})$. The order of H is 29120.
There is one serial 7-block and six simple 7-blocks of H.

$\circ \;\text{———}\; \bullet_{m=3} \;\text{———}\; \circ$

Let $G = 2.Sz(8)$, the double cover of $Sz(8)$. Then the principal block of G is serial, but there is a non-serial block.

$\circ \;\text{———}\; \bullet_{m=3} \;\text{———}\; \circ$

$\circ \;\text{———}\; \circ \;\text{———}\; \bullet_{m=3}$

Proposition 14

If $\mathrm{char} F = 7$*, then the ring FH is serial, but FG is not serial.*

Open problems

Problem 1. To find all pairs (F, G), where F is a field, and G is a finite group, such that the group ring FG is serial.

Problem 2. To find all pairs (p, G), where p is a prime number, and G is a finite group, such that the Brauer tree of each p-block of G is a star.

Problem 3. To find all pairs (S, G), where S is a ring, and G is a finite group, such that the group ring SG is serial.

THANKS FOR YOUR ATTENTION