CLOSED SETS OF FINITARY FUNCTIONS BETWEEN FINITE FIELDS OF COPRIME ORDER.

Stefano Fioravanti
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Institute for Algebra
Austrian Science Fund FWF P29931

JOHANNES KEPLER UNIVERSITY LINZ

$(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids

Definition

Let p and q be powers of prime numbers. A $(\mathbb{F}_p, \mathbb{F}_q)$**-linear closed clonoid** is a non-empty subset C of $\bigcup_{n \in \mathbb{N}} \mathbb{F}_p^{\mathbb{F}_q^n}$ such that:

(1) if $f, g \in C^{[n]}$ then:

$$f +_p g \in C^{[n]};$$

(2) if $f \in C^{[m]}$ and $A \in \mathbb{F}_q^{m \times n}$ then:

$$g : (x_1, ..., x_n) \mapsto f(A \cdot_q (x_1, ..., x_n)^t) \text{ is in } C^{[n]}.$$

Known results

[1] E. Aichinger, P. Mayr, Polynomial clones on groups of order pq, in: Acta Mathematica Hungarica, Volume 114, Number 3, Page(s) 267-285, 2007.
(All 17 clones containing $(\mathbb{Z}_p \times \mathbb{Z}_q, +, (1,1))$);

[2] J. Bulín, A. Krokhin, and J. Opršal, Algebraic approach to promise constraint satisfaction, arXiv:1811.00970, 2018.

[3] S. Kreinecker, Closed function sets on groups of prime order, Manuscript, arXiv:1810.09175, 2018.
(All finitely many clones containing $(\mathbb{Z}_p, +)$).

$(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids

Proposition

The intersection of $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids is again a $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid.

Definition

Let K be subset of n-ary function from $\mathbb{F}_q$ to $\mathbb{F}_p$ and $\mathcal{A}$ the set of all the $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids. We define the **$(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid generated by K** as:

$$C^{(p,q)}(K) = \bigcap_{\mathcal{C} \in \mathcal{B}} \mathcal{C}$$

where $\mathcal{B} = \{\mathcal{C} | \mathcal{C} \in \mathcal{A}, K \subseteq \mathcal{C}\}$.

Definition

Let f be an n-ary function from a group **G**$_1$ to a group **G**$_2$. We say that f is 0**-preserving** if:

$$f(0_{G_1}, ..., 0_{G_1}) = 0_{G_2}.$$

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Remark

Let p and q be prime numbers. The 0-preserving functions from $\mathbb{F}_q$ to $\mathbb{F}_p$ form a $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid.

Other examples

(1) The constant functions.

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Definition

Let f be a function from $\mathbb{F}_q^n$ to $\mathbb{F}_p$. The function f is a **star** function if and only if for every vector **w** $\in \mathbb{F}_q^n$ there exists $k \in \mathbb{F}_p$ such that for every $\lambda \in \mathbb{F}_q - \{0\}$:

$$f(\lambda \mathbf{w}) = k.$$

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(3) The star functions (functions constant on rays from the origin, but not in the origin).

A characterization

Theorem (SF)

Let p and q be powers of different primes. Then every $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid C is generated by its unary functions. Thus $C = C^{(p,q)}(C^{[1]})$.

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Corollary

Let p and q be two distinct prime numbers. Then every $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid has a set of finitely many unary functions as generators. Hence there are only finitely many distinct $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids.

Example

Matrix representation of a function $f : \mathbb{Z}_5^2 \mapsto \mathbb{Z}_{11}$

$$f(i,j) = a_{ij} \text{ where } (a_{ij}) \in \mathbb{Z}_{11}^{5\times 5}$$

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$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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The function $f_1 : \mathbb{Z}_5 \mapsto \mathbb{Z}_{11}$ defined as $f_1(0) = 0$, $f_1(i) = a_i$ for $i = 1, \ldots, 4$ is in $C(\{f\})$.

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How to generate f from f_1 in a $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid?

Example

Let us define the function $s : \mathbb{Z}_5^2 \mapsto \mathbb{Z}_{11}$:

$$s(x, y) = f_1(y)$$

$$\begin{pmatrix} a_4 & a_4 & a_4 & a_4 & a_4 \\ a_3 & a_3 & a_3 & a_3 & a_3 \\ a_2 & a_2 & a_2 & a_2 & a_2 \\ a_1 & a_1 & a_1 & a_1 & a_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

Let us define the function $s : \mathbb{Z}_5^2 \mapsto \mathbb{Z}_{11}$:

$$s(x, y) = f_1(y) + f_1(y - x)$$

$$\begin{pmatrix} 2a_4 & a_3 + a_4 & a_2 + a_4 & a_1 + a_4 & a_4 \\ 2a_3 & a_2 + a_3 & a_1 + a_3 & a_3 & a_3 + a_4 \\ 2a_2 & a_1 + a_2 & a_2 & a_2 + a_4 & a_2 + a_3 \\ 2a_1 & a_1 & a_1 + a_4 & a_1 + a_3 & a_1 + a_2 \\ 0 & a_4 & a_3 & a_2 & a_1 \end{pmatrix}$$

Example

Let us define the function $s : \mathbb{Z}_5^2 \mapsto \mathbb{Z}_{11}$:

$$s(x, y) = f_1(y) + f_1(y - x) + f_1(y - 2x)$$

$$\begin{pmatrix} 3a_4 & a_2 + a_3 + a_4 & a_2 + a_4 & a_1 + a_3 + a_4 & a_1 + a_4 \\ 3a_3 & a_1 + a_2 + a_3 & a_1 + a_3 + a_4 & a_2 + a_3 & a_3 + a_4 \\ 3a_2 & a_1 + a_2 & a_2 + a_3 & a_1 + a_2 + a_4 & a_2 + a_3 + a_4 \\ 3a_1 & a_1 + a_4 & a_1 + a_2 + a_4 & a_1 + a_3 & a_1 + a_2 + a_3 \\ 0 & a_3 + a_4 & a_1 + a_3 & a_2 + a_4 & a_1 + a_2 \end{pmatrix}$$

Example

Let us define the function $s : \mathbb{Z}_5^2 \mapsto \mathbb{Z}_{11}$:

$$s(x, y) = f_1(y) + f_1(y - x) + f_1(y - 2x) + f_1(y - 3x) + f_1(y - 4x)$$

$$\begin{pmatrix} 5a_4 & \lambda & \lambda & \lambda & \lambda \\ 5a_3 & \lambda & \lambda & \lambda & \lambda \\ 5a_2 & \lambda & \lambda & \lambda & \lambda \\ 5a_1 & \lambda & \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda & \lambda & \lambda \end{pmatrix}$$

where $\lambda = a_1 + a_2 + a_3 + a_4$

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Let us define the function $s : \mathbb{Z}_5^2 \mapsto \mathbb{Z}_{11}$:

$$s(x,y) = f_1(y) + f_1(y-x) + f_1(y-2x) + f_1(y-3x) + f_1(y-4x) - \sum_{k=1}^{q-1} f_1(kx)$$

$$\begin{pmatrix} 5a_4 & 0 & 0 & 0 & 0 \\ 5a_3 & 0 & 0 & 0 & 0 \\ 5a_2 & 0 & 0 & 0 & 0 \\ 5a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

Let $n \in \mathbb{N}$ be s.t $n * 5 \equiv_{11} 1$. Hence $n * s$ is the function:

$$\begin{pmatrix} a_4 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

for all $x, y \in \mathbb{Z}_5$:

$$f(x, y) = n * s(x - y, y)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition

Let $\mathbb{F}_q$ and $\mathbb{F}_p$ be finite fields and let $f : \mathbb{F}_q \to \mathbb{F}_p$ be a unary function. Let α be a generator of the multiplicative subgroup $\mathbb{F}_q^\times$ of $\mathbb{F}_q$. We define the α-vector encoding of f as the vector $\mathbf{v} \in \mathbb{F}_p^q$ such that:

$$
\begin{aligned}
v_{i+1} &= f(\alpha^i) \quad \text{for} \quad 0 \leq i \leq q-2, \\
v_0 &= f(0).
\end{aligned}
$$

How to describe the unary functions

Proposition

Let C be a $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid, and let α be a generator of the multiplicative subgroup $\mathbb{F}_q^{\times}$ of $\mathbb{F}_q$. Then the set S of all the α-vector encodings of unary functions in C is a subspace of $\mathbb{F}_p^q$ and it satisfies:

$$(x_0, x_k, x_{k+1}, \ldots, x_{q-1}, x_1, \ldots, x_{k-1}) \in S \quad (1)$$

and

$$(x_0, \ldots, x_0) \in S \quad (2)$$

for all $(x_0, \ldots, x_{q-1}) \in S$ and $k \in \{1, \ldots, q-1\}$.

A characterization

We denote by $A(p,q)$ and $C(p,q)$ respectively the linear transformations of $\mathbb{F}_p^q$ defined as:

$$A(p,q)((v_0,\ldots,v_{q-1})) = (v_0, v_k, v_{k+1},\ldots,x_{q-1},v_1,\ldots,v_{k-1})$$
$$C(p,q)((v_0,\ldots,v_{q-1})) = (v_0,\ldots,v_0)$$

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An **invariant subspace** of a linear operator T on some vector space V is a subspace W of V that is preserved by T; that is, $T(W) \subseteq W$.

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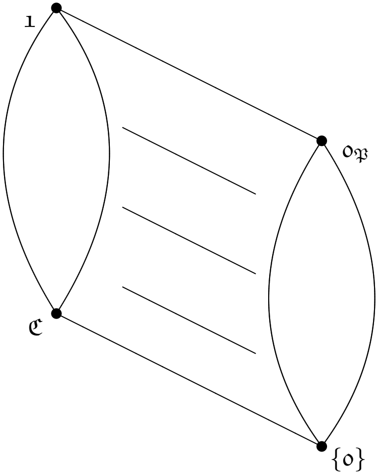
Theorem (SF)

Let p and q be powers of distinct prime numbers. Then the lattice of all $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids $\mathbf{L}(p,q)$ is isomorphic to the lattice $\mathbf{L}(A(p,q), C(p,q))$ of all the $(A(p,q), C(p,q))$-invariant subspaces of $\mathbb{F}_p^q$.

Lattice of the $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids



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Let us denote by $\mathbf{2}$ the two-elements chain and, in general, by $\mathbf{C}_k$ the chain with k elements.

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Theorem (SF)

Let p and q be powers of different primes. Let $\prod_{i=1}^n p_i^{k_i}$ be the prime factorization of the polynomial $g = x^{q-1} - 1$ in $\mathbb{F}_p[x]$. Then the number of distinct $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids is $2\prod_{i=1}^n (k_i + 1)$ and the lattice of all the $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids, $\mathbf{L}(p, q)$, is isomorphic to

$$\mathbf{2} \times \prod_{i=1}^n \mathbf{C}_{k_i+1}.$$

Lattice of the $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids

Corollary

Let p and q be powers of distinct primes. Then the lattice $\mathbf{L}(p, q)$ of the $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoids is a distributive lattice.

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Every 0-preserving $(\mathbb{F}_p, \mathbb{F}_q)$-linear closed clonoid is principal (generated by a unary functions).

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THANK YOU!!!!