Topology on a BE-algebra via Right Application of BE-ordering

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Preliminaries

Definition [1]

An algebra $(X; *, 1_X)$ is called a *BE-algebra* if the following hold: for all $x, y, z \in X$,

(BE1) $x * x = 1_X$
(BE2) $x * 1_X = 1_X$
(BE3) $1_X * x = x$
(BE4) $x * (y * z) = y * (x * z)$.

A relation "$\leq$" on X, called *BE-ordering*, is defined by $x \leq y$ if and only if $x * y = 1_X$.

Example [3]

Let $N_0 = \mathbb{N} \cup \{0\}$ and let $*$ be the binary operation on N_0 defined by

$$x * y = \begin{cases} 0 & \text{if } y \leq x \\ y - x & \text{if } x < y. \end{cases}$$

Then $(N_0; *, 0)$ is a BE-algebra where $1_{N_0} = 0$.

- A BE-algebra $(X, *, 1_X)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

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- A BE-algebra $(X, *, 1_X)$ is said to be *transitive* if it satisfies the condition: $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$.

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- A BE-algebra $(X, *, 1_X)$ is said to be *transitive* if it satisfies the condition: $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$.
- A BE-algebra $(X, *, 1_X)$ is said to be *commutative* if it satisfies $(x * y) * y = (y * x) * x$ for all $x, y \in X$.

- An element $a \neq 1_X$ of a BE-algebra $(X, *, 1_X)$ is said to be a *dual atom* of X if $a \leq x$ implies either $a = x$ or $x = 1_X$ for all $x \in X$.

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- A BE-algebra $(X, *, 1_X)$ is called *dual atomistic* if every non-unit element of X is a dual atom in X.
- Denote by $\mathcal{A}(X)$ the set of all dual atoms of X.

Some Properties of r_X

Definition

Let X be a BE-algebra. For any $A \subseteq X$, the set

$$r_X(A) = \{x \in X \mid a * x = 1_X, \forall a \in A\}$$

is called the *set induced by right ordering on* A.

Let A and B be subsets of X. Then the following hold:

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- $r_X(\varnothing) = X$.
- If $A \subseteq B$, then $r_X(B) \subseteq r_X(A)$.
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- If $1_X \in A$, then $r_X(A) = \{1_X\}$.
- Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of X, then $\bigcap_{\alpha \in I} r_X(A_\alpha) = r_X\left(\bigcup_{\alpha \in I} A_\alpha\right)$.

A Basis $\mathcal{B}_r(X)$ for a topology on X

Theorem

Let X be a BE-algebra. Then

$$\mathcal{B}_r(X) = \{r_X(A) : \varnothing \neq A \subseteq X\}$$

is a basis for some topology on X.

We denote by $\tau_r(X)$ the topology generated by $\mathcal{B}_r(X)$. Consider $N_0 = \mathbb{N} \cup \{0\}$ and $*$ defined by

$$x * y = \begin{cases} 0 & \text{if } y \leq x \\ y - x & \text{if } x < y. \end{cases}$$

Let $z \in N_0$. Then $r_{N_0}(z) = \{0, 1, 2, \ldots, z\}$. It is easy to see that $\mathcal{B} = \{r_{N_0}(z) : z \in N_0\} \subseteq \mathcal{B}_r(N_0)$. Suppose that $\varnothing \neq A \subseteq N_0$ and $w = \min A$. Then $r_{N_0}(A) = \bigcap_{a \in A} r_{N_0}(\{a\})$. It follows that $r_{N_0}(A) = \{0, 1, 2, \ldots, w\} = r_{N_0}(w) \in \mathcal{B}$. Hence, $\mathcal{B}_r(N_0) = \mathcal{B} = \{r_{N_0}(z) : z \in N_0\}$. Let $\varnothing \neq G \in \tau_r(N_0)$. Then $G = \bigcup_{x \in K} r_{N_0}(x)$ for some $\varnothing \neq K \subseteq N_0$. Clearly, $K \subseteq G$. Suppose first that $|G| < \infty$ and let $v = \max K$. Then $G = r_{N_0}(v)$.

Next, suppose that G is an infinite set. Suppose further that $G \neq N_0$, say $m \in N_0 \setminus G$. Then $m \notin r_{N_0}(x)$ for all $x \in K$. This implies that $x < m$ for all $x \in K$. Hence, $G \subseteq r_{N_0}(m)$, contrary to the assumption that G is an infinite set. Therefore, $G = N_0$. Accordingly, $\tau_r(N_0) = \{\varnothing, N_0\} \cup \{r_{N_0}(z) : z \in N_0\} = \{\varnothing, N_0\} \cup \mathcal{B}_r(N_0)$.

Theorem

Let X be a BE-algebra. Then $(X, \tau_r(X))$ is connected.

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Theorem

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If X is a finite BE-algebra, then

$$\mathcal{S}_r(X) = \{r_X(\{a\}) : a \in X\}$$

is a subbase of $\tau_r(X)$.

A Basis $\mathcal{B}_r(X)$ for a topology on X

Theorem

Let X be a BE-algebra with $|X| \geq 2$. Then

$$\mathcal{B}_r(X) = \{\{1_X, a\} : a \in \mathcal{A}(X)\} \bigcup \{r_X(A) : \ A \cap \mathcal{A}(X) = \varnothing\}.$$

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Corollary

Let X be a BE-algebra with $|X| \geq 2$. If $\mathcal{A}(X) = \{a\}$, then

$$\mathcal{B}_r(X) = \{\{1_X, a\}\} \cup \{r_X(A) : \; a \notin A\}.$$

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Theorem

Let X be a BE-algebra with $|X| \geq 2$. Then $\mathcal{B}_r(X) = \{\{1_X\}\} \cup \{\{1_X, a\} : a \in X \setminus \{1_X\}\}$ if and only if X is dual atomistic.

This section gives some characterizations of the elementary concepts associated with the topological space $(X, \tau_r(X))$.

Theorem

Let X be a BE-algebra with $|X| \geq 2$. Then $\tau_r(X)$ is the particular point 1_X topology τ_{1_X} on X if and only if X is dual atomistic.

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Let S be a subalgebra of a transitive BE-algebra X with $|S| \geq 2$. Then $\tau_r(S)$ coincides with the relative topology τ_S on S.

In a dual atomistic BE-algebra X with respect to $\tau_r(X)$, every set that contains 1_X is open and every set that does not contain 1_X is closed. Hence, the following corollary is true.

Corollary

Let X be a dual atomistic BE-algebra with $|X| \geq 2$ and let $O, C \subseteq X$. Then with respect to $\tau_r(X)$, we have

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Let X be a dual atomistic BE-algebra with $|X| \geq 2$ and let $O, C \subseteq X$. Then with respect to $\tau_r(X)$, we have

(i)

$$Int(O) = \begin{cases} \varnothing & \text{if } 1_X \notin O \\ O & \text{if } 1_X \in O, \end{cases} \qquad \text{and}$$

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$$Int(O) = \begin{cases} \varnothing & \text{if } 1_X \notin O \\ O & \text{if } 1_X \in O, \end{cases} \qquad \text{and}$$

(ii)

$$\overline{C} = \begin{cases} X & \text{if } 1_X \in C \\ C & \text{if } 1_X \notin C. \end{cases}$$

Theorem

Let X be a BE-algebra and let $D \subseteq X$. Then with respect to $\tau_r(X)$, we have

Theorem

Let X be a BE-algebra and let $D \subseteq X$. Then with respect to $\tau_r(X)$, we have

(i) $z \in Int(D)$ if and only if there exists $\varnothing \neq B \subseteq X$ such that $b * z = 1_X$ for all $b \in B$ and for all $x \in X$, $x \in D$ whenever $b * x = 1_X$ for all $b \in B$.

Characterizations Involving the Topology $\tau_r(X)$

Theorem

Let X be a BE-algebra and let $D \subseteq X$. Then with respect to $\tau_r(X)$, we have

(i) $z \in Int(D)$ if and only if there exists $\varnothing \neq B \subseteq X$ such that $b * z = 1_X$ for all $b \in B$ and for all $x \in X$, $x \in D$ whenever $b * x = 1_X$ for all $b \in B$.

(ii) $y \in \overline{D}$ if and only if for each $\varnothing \neq A \subseteq X$ with $a * y = 1_X$ for all $a \in A$, there exists $d \in D$ such that $a * d = 1_X$ for all $a \in A$.

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(iii) D is dense in X if and only if $1_X \in D$. In particular, $\{1_X\}$ is dense in X.

Theorem

Let $(X_1, *_{X_1}, 1_{X_1})$ and $(X_2, *_{X_2}, 1_{X_2})$ be BE-algebras. Then a function $f : (X_1, \tau_r(X_1)) \to (X_2, \tau_r(X_2))$ is continuous on X_1 if and only if for each $B \subseteq X_2$ and for each $x \in X_1$ such that $b \leq f(x)$ for all $b \in B$, there exists $A \subseteq X_1$ satisfying the following conditions:

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(i) $a \leq x$ for all $a \in A$

(ii) $b \leq f(z)$ for all $b \in B$ whenever $a \leq z$ for all $a \in A$.

Theorem

Let $(X_1, *_{X_1}, 1_{X_1})$ and $(X_2, *_{X_2}, 1_{X_2})$ be BE-algebras and let $f : (X_1, \tau_r(X_1)) \to (X_2, \tau_r(X_2))$ be a function. Then

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(i) f is open if and only if for each $A \subseteq X_1$ and for each $x \in X_1$ with $a \leq x$ for all $a \in A$, there exists $B \subseteq X_2$ satisfying the following properties:

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(a) $b \leq f(x)$ for all $b \in B$

(b) there exists $z \in X_1$ with $a \leq z$ for all $a \in A$ and $f(z) = y$ whenever $a \leq f^{-1}(y)$ for all $a \in A$ and $b \leq y$ for all $b \in B$.

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(ii) f is closed if and only if for each $\tau_r(X_1)$-closed set F and for all $y \in X_2$ with $y \neq f(x)$ for all $x \in F$, there exists $A_y \subseteq X_2$ such that $r_{X_2}(A_y) \cap f(F) = \varnothing$ and $a \leq y$ for all $a \in A_y$.

Thank you for listening!!

(Proverbs 3:5) Trust in the Lord with all of your heart and lean not on your own understanding.

God bless us all. :)

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