ON COLEMAN AUTOMORPHISMS OF FINITE GROUPS AND THEIR MINIMAL NORMAL SUBGROUPS

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DEFINITIONS

Definition

Let G be a finite group and $\sigma \in \mathrm{Aut}(G)$. If for any prime p dividing the order of G and any Sylow p-subgroup P of G, there exists a $g \in G$ such that $\sigma|_P = \mathrm{conj}(g)|_P$, then σ is said to be a Coleman automorphism.

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Denote $\mathrm{Aut}_{col}(G)$ for the set of Coleman automorphisms, $\mathrm{Inn}(G)$ for the set of inner automorphisms and set

$$\mathrm{Out}_{col}(G) = \mathrm{Aut}_{col}(G)/\mathrm{Inn}(G).$$

ELEMENTARY RESULTS

Theorem (Hertweck and Kimmerle)

Let G be a finite group. The prime divisors of $|\mathrm{Aut}_{col}(G)|$ are also prime divisors of $|G|$.

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Lemma

Let N be a normal subgroup of a finite group G. If $\sigma \in \mathrm{Aut}_{col}(G)$, then $\sigma|_N \in \mathrm{Aut}(N)$.

NORMALIZER PROBLEM

Let G be a group and R a ring (denote $U(RG)$ for the units of RG), then we clearly have that

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Is this an equality?

$$N_{U(RG)}(G) = GC_{U(RG)}(G)$$

Of special interest: $R = \mathbb{Z}$

SETUP FOR REFORMULATION

If $u \in N_{U(RG)}(G)$, then u induces an automorphism of G:

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Denote $\mathrm{Aut}_U(G; R)$ for the group of these automorphisms ($\mathrm{Aut}_U(G)$ if $R = \mathbb{Z}$) and $\mathrm{Out}_U(G; R) = \mathrm{Aut}_U(G; R)/\mathrm{Inn}(G)$

REFORMULATION

Theorem (Jackowski and Marciniak)

G a finite group, R a commutative ring. TFAE

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Theorem

Let G be a finite group. Then,

$$\mathrm{Aut}_U(G) \subseteq \mathrm{Aut}_{col}(G)$$

MORE KNOWN RESULTS

Lemma (Hertweck)

Let p be a prime number and $\alpha \in \mathrm{Aut}(G)$ of p-power order. Assume that there exists a normal subgroup N of G, such that $\alpha|_N = \mathrm{id}_N$ and α induces identity on G/N. Then α induces identity on $G/O_p(Z(N))$. Moreover, if α also fixes a Sylow p-subgroup of G elementwise, i.e. α is p-central, then α is conjugation by an element of $O_p(Z(N))$.

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Theorem (Hertweck and Kimmerle)

For any finite simple group G, there is a prime p dividing $|G|$ *such that p-central automorphisms of G are inner automorphisms.*

SELF-CENTRALITY

Theorem (A.V.A.)

Let G be a normal subgroup of a finite group K. Let N be a minimal non-trivial characteristic subgroup of G. If $C_K(N) \subseteq N$, then every Coleman automorphism of K is inner. In particular, the normalizer problem holds for these groups.

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Proof.

1. N is characteristically simple
2. Restrict to simple groups
3. Subdivide Abelian vs. Non-abelian
4. Hertweck's result

$\square$

COROLLARY 1

Corollary

Let $G = P \rtimes H$ be a semidirect product of a finite p-group P and a finite group H. If $C_G(P) \subseteq P$, then G has no non-inner Coleman automorphisms.

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The wreath product $G \wr H$ of G, H is defined as $\Pi_{h \in H} G \rtimes H$, where H acts on the indices.

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The wreath product $G \wr H$ of G, H is defined as $\prod_{h \in H} G \rtimes H$, where H acts on the indices.

Corollary

Let S be a finite simple group, I a finite set of indices, and H any finite group. If $G = \prod_{i \in I} S \rtimes H$ is a group such that $C_G(\prod_{i \in I} S) \subseteq Z(\prod_{i \in I} S)$. Then G has no non-inner Coleman automorphism. In particular, the normalizer problem holds for G. Then, in particular, the wreath product $S \wr H$ has no non-inner Coleman automorphisms. Moreover, in both cases the normalizer problem holds for G.

QUESTIONS OF HERTWECK AND KIMMERLE

Questions (Hertweck and Kimmerle)

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Partial Answer (Hertweck, Kimmerle)

Besides giving several conditions, all three statements hold if G *is assumed to be a* p*-constrained group.*

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Partial Answers (Van Antwerpen)

1. *No new result.*
2. *True, if $O_p(G) = O_{p'}(G) = 1$, where p is an odd prime and the order of every direct component of $E(G)$ is divisible by p.*
3. *True, if the unique minimal non-trivial normal subgroup is non-abelian. True, if question 2 has a positive answer.*

FURTHER RESEARCH

Inkling of Idea

In case G has a unique minimal non-trivial normal subgroup N, which we may assume to be abelian, we may be able to use the classification of the simple groups and the list of all Schur multipliers to give a technical proof.

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Related Question

Gross conjectured that for a finite group G with $O_{p'}(G) = 1$ for some odd prime p, p-central automorphisms of p-power order are inner. Hertweck and Kimmerle believed this is possible using the classification of Schur mutipliers.

FURTHER RESEARCH 2

Stretch of Idea

M. Murai connected Coleman automorphisms of finite groups to the theory of blocks in representation theory, showing several slight generalizations of known theorems. Looking into his methods may prove useful.

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