A still undecided point on groups with an identity

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To the organizers

A G T A

Ancora Grazie, Thanks Again!

The Burnside problems originated from the famous paper of W. Burnside: *On an unsettled question in the theory of discontinuous groups*, Quart. J. Pure Appl. Math. **33** (1902).

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General Burnside Problem . . . . . .

"A still undecided point"

Is a finitely generated periodic group necessarily finite?

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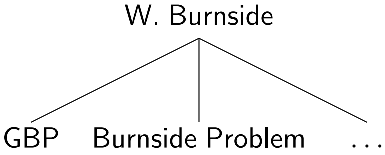
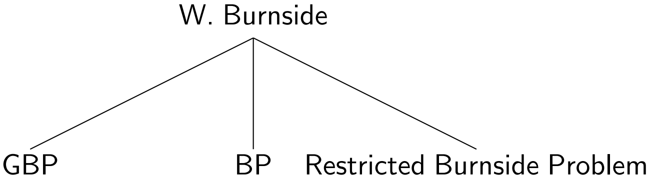
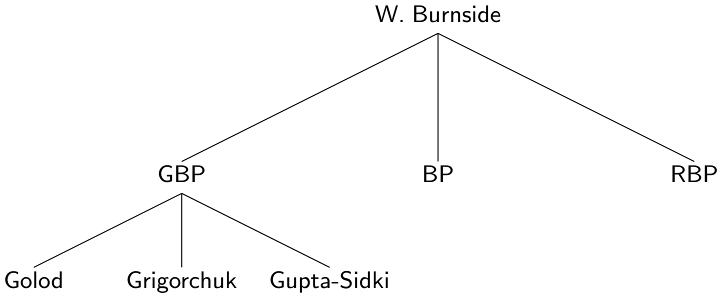


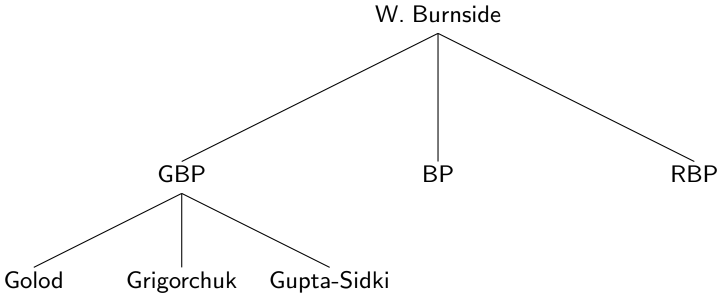
Is a finitely generated group of finite exponent necessarily finite?

After some partial results, in the early 1930's, the original questions were replaced with the so-called restricted Burnside problem.



Is the order of a d-generator finite group of exponent n bounded in terms of d and n only?


W. Burnside
GBP
BP
RBP
Golod
Grigorchuk
Gupta-Sidki



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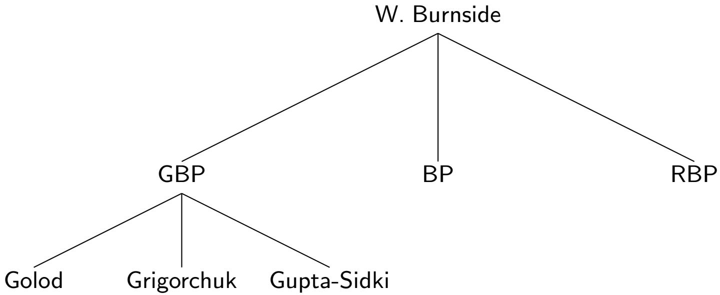

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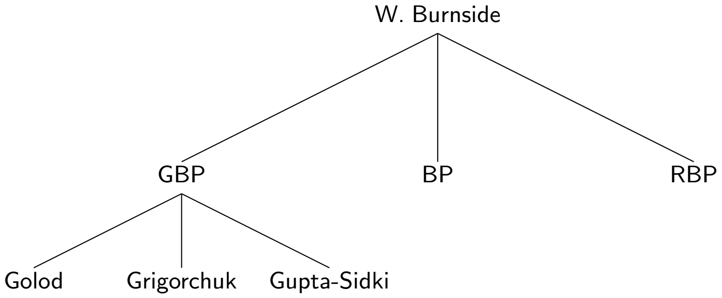

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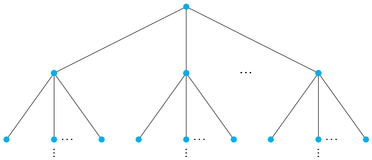

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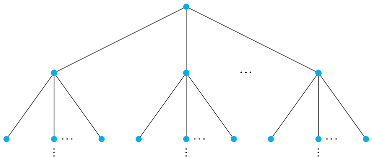

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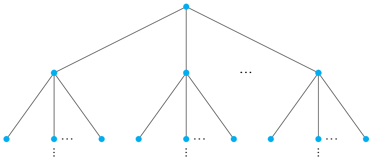

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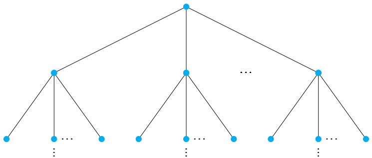

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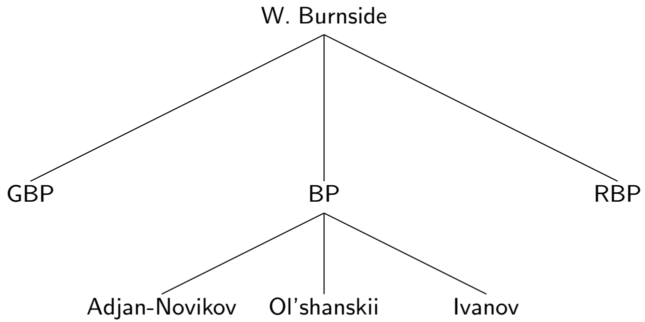
The set $Aut\,\mathcal{T}$ is a group with respect to composition.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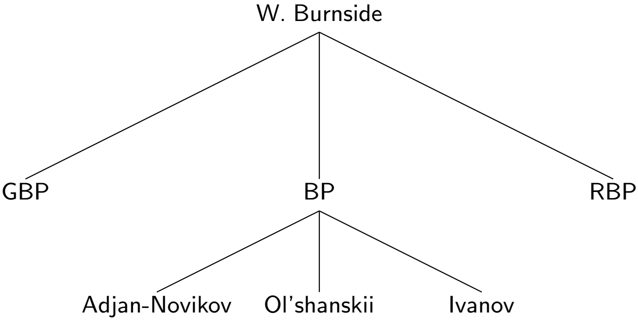
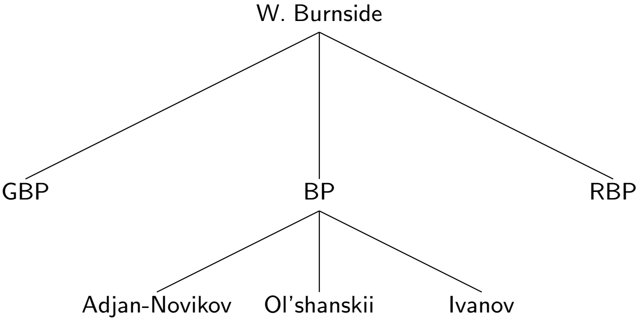


Let $B(d,n) = F/F^n$ be the free Burnside group of rank d and exponent n, where F is a free group with d generators.



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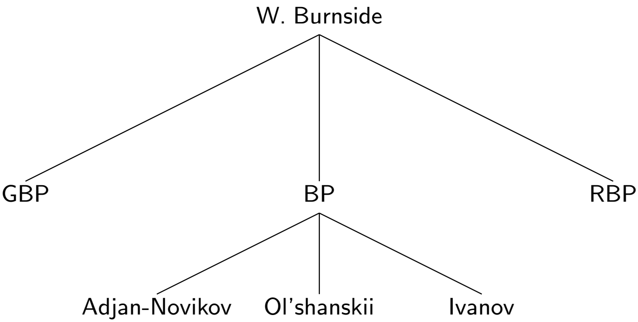

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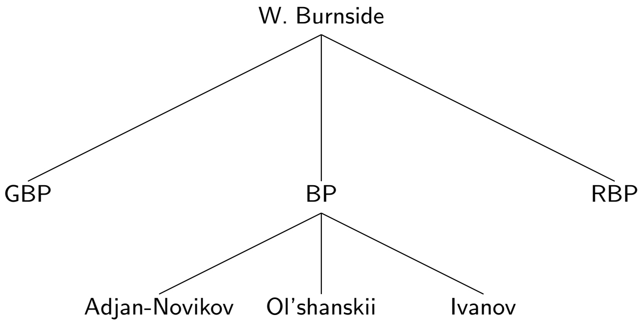

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- $B(d, n)$ is infinite for n odd, $n \geq 4381$ (Adjan-Novikov, 1968);
- For any prime $p > 10^{75}$, there exists an infinite simple 2-generator group whose of all proper subgroups are cyclic of order p (Ol'shanskii, 1982);
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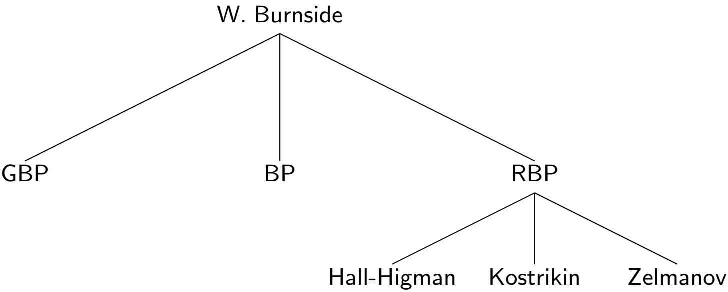
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Hall-Higman
Kostrikin
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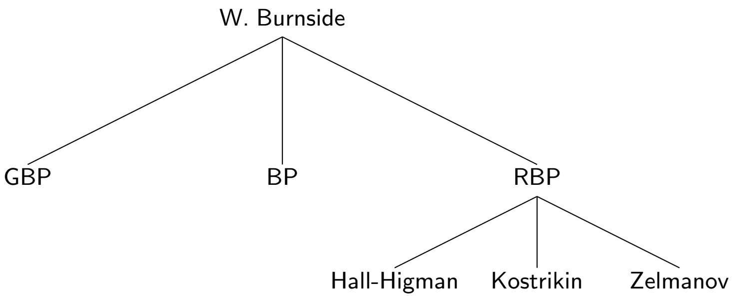
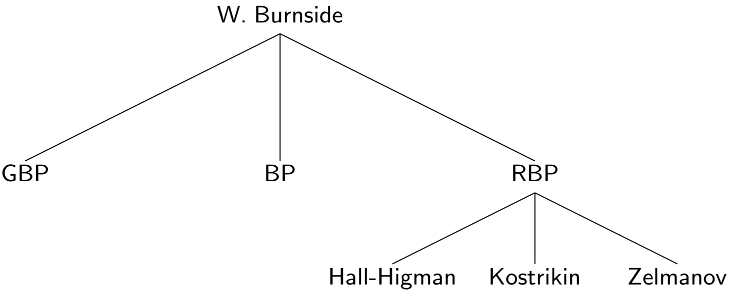


By the reduction theorem of P. Hall and G. Higman, and the classification of finite simple groups, the exponent of the group can be assumed to be a power of a prime.



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A positive solution of the RBP was obtained for prime exponent by A.I. Kostrikin and for any prime-power exponent by E.I. Zelmanov applying their theorems on Engel Lie algebras.

Consequence of the positive solution of the RBP

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Let $L_p(G)$ be the Lie algebra associated with the Zassenhaus-Jennings-Lazard series

$$G = D_1 \geq D_2 \geq \dots$$

where

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In Zelmanov's solution of the RBP, L satisfies the linearized Engel identity $\sum_{\sigma \in Sym(k)} ad(x_{\sigma_{(1)}}) \ldots ad(x_{\sigma_{(k)}}) = 0$ and each Lie-commutator in $a_1, a_2, \ldots, a_m$ is ad-nilpotent of index at most n.

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For an "answer" to (e2) see the video of E. Rips at
`www.sites.google.com/site/geowalks2014/home/workshop`

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There is an orderable Engel group which is not locally nilpotent.

A group G is *orderable* if there exists a full order relation $\leq$ on the set G such that

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Open question

Find a simpler example of a (residually nilpotent) Engel group which is not locally nilpotent.

The Grigorchuk group

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The proof relies on a computer calculation by GAP.

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Wait for the answer!

Marialaura's talk: 10:55 - 11:20.

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A group G is a *nil group* if all its elements are bounded Engel, i.e., for any $g \in G$ there is $n = n(g) \geq 1$ such that $[x,_n g] = 1$ for all $x \in G$.

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- if G is locally graded, then G is locally (nilpotent-by-finite);
- if G is orderable, then G is nilpotent.

Furthermore, the class of all groups G in which the w-values are n-Engel and the verbal subgroup $w(G)$ is locally nilpotent is a variety.

References

- *Bounded Engel elements in groups satisfying an identity* (with R. Bastos, N. Mansuroglu and M. Tota), submitted
- *On varieties of groups satisfying an Engel type identity* (with P. Shumyatsky and M. Tota), J. Algebra **447** (2016), 479–489
- *On locally graded groups with a word whose values are Engel* (with P. Shumyatsky and M. Tota), Proc. Edinburgh Math. Soc., Proceedings of the Edinburgh Mathematical Society **59** (2016), 533–539
- *An Engel condition for orderable groups* (with P. Shumyatsky and M. Tota), Bull. Braz. Math. Soc. (N.S.), Bull. Braz. Math. Soc. (N.S.) **46** (2015), 461–468
- *On groups admitting a word whose values are Engel* (with R. Bastos, P. Shumyatsky and M. Tota), Int. J. Algebra Comput. **23** no. 1 (2013), 81–89

To the audience

Thanks for your attention!