Semi-braces and the Yang-Baxter equation

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Definition

A linear map $R : V \otimes V \longrightarrow V \otimes V$ is called a **Yang-Baxter operator** *if it is a solution of the quantum Yang-Baxter equation, i.e.,*

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \tag{1}$$

holds in the monoid of the linear maps of $V \otimes V \otimes V$ in itself.

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In particular, the following result is useful:

Proposition

If B is a non-empty set, denoted by $\tau : B \times B \to B \times B$ the twist map, i.e., $\tau(x, y) = (y, x)$, we have that $\mathcal{R} : B \times B \to B \times B$ is a Yang-Baxter map if and only if the function $r := \tau\mathcal{R} : B \times B \to B \times B$ satisfies the **braid equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2, \tag{2}$$

where $r_1 := r \times id_B$ and $r_2 := id_B \times r$.

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Usually, r or $\mathcal{R}$ is called a **set-theoretic solution of the quantum Yang-Baxter equation**.

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- **idempotent** *if $r^2 = r$.*

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- [V.V. Lyubashenko] If B is a set, the function $r : B \times B \to B \times B$ given by

$$r(x, y) = (f(y), g(x)),$$

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▷ If we fix $c \in B$ and consider $f : B \to B$ defined by $f(x) = c$, for every $x \in B$, and $g = f$, then we obtain $r(x, y) = (c, c)$, for all $x, y \in B$.

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$$r^2(x, y) = r(c, c) = (c, c) = r(x, y),$$

for all $x, y \in B$, then r is idempotent.

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- [GU Pei, 1997] If B is a group, the function $r : B \times B \to B \times B$ defined by

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- Since $\rho_y = f$, for every $y \in B$, we have that r is right non-degenerate if and only if f is bijective.

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- [B.B. Venkov] If B is a set, the function $r : B \times B \to B \times B$, defined by

$$r\left(x, y\right) = \left(x * y, x\right),$$

for all $x, y \in B$, is a solution where $*$ is a self-distributive operation on B, i.e.,

$$x * \left(y * z\right) = \left(x * y\right) * \left(x * z\right), \tag{4}$$

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Such a structure $(B, *)$ is called *shelf* or *self-distributive structure* and, nowadays, receives attention by some authors, as for instance Lebed and Vendramin [2017].

Lebed solutions

- [Lebed, 2016] Let $(B, \vee, \wedge)$ be a distributive lattice. Then the function $r : B \times B \to B \times B$ defined by

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  In particular, r is neither right nor left non-degenerate and since

  $$\begin{aligned} r^2\,(x, y) &= r\,(x \wedge y, x \vee y) = ((x \wedge y) \wedge (x \vee y)\,, (x \wedge y) \vee (x \vee y)) \\ &= (x \wedge y, x \vee y) = r\,(x, y)\,, \end{aligned}$$

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Currently, some authors are interested in finding **degenerate solutions** and **idempotent solutions** such as, for instance, Lebed and Vendramin.

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A very brief state of the art

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We will focus on solutions that are only **left non-degenerate**. In particular, we will show how to determine such solutions through a new structure, the **semi-brace**. We will see that we obtain also **idempotent** solutions.

Semi-braces

In [*Semi-braces and the Yang-Baxter equation*, J. Algebra **483** (2017), 163–187], F. Catino, I. Colazzo and myself introduce the *semi-brace*, a structure that allows us to obtain new solutions of the Yang-Baxter equation.

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Definition

A set B with two operations $+$ and $\circ$ is a **(left) semi-brace** *if $(B, +)$ is a left cancellative semigroup, $(B, \circ)$ is a group and*

$$a \circ (b + c) = a \circ b + a \circ \left(a^{-} + c\right) \qquad (5)$$

holds for all $a, b, c \in B$, where by a^{-} we denote the inverse of the element a with respect to $\circ$.

How to obtain a solution through a semi-brace

Theorem (Catino, Colazzo, P.S., J. Algebra, 2017)

Let B be a semi-brace. Then, the function $r : B \times B \to B \times B$ given by

$$r\left(a, b\right) = \left(\, a \circ \left(a^{-} + b\right),\ \left(a^{-} + b\right)^{-} \circ b \,\right) \tag{6}$$

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Therefore, **every solution associated to a semi-brace B that is not a skew brace is left non-degenerate and right degenerate**.

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Therefore, **every solution associated to a semi-brace B that is not a skew brace is left non-degenerate and right degenerate**. Clearly, this solution is not bijective.

Example

Let $(B, \circ)$ be a group, f an endomorphism of $(B, \circ)$ such that $f^2 = f$ and $(B, +, \circ)$ the semi-brace where $a + b = b \circ f(a)$, for all $a, b \in B$.

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  $$r^2(a, b) = r(a \circ b, 0) = (a \circ b \circ 0, 0) = (a \circ b, 0) = r(a, b)$$

  and so r is idempotent.

The solution associated to a skew brace

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Let $(B,+,\circ)$ be a skew brace. Then, the function $r : B \times B \to B \times B$ given by

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for all $a, b \in B$, is a non-degenerate bijective solution of the Yang-Baxter equation, where if $a \in B$, then

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Hence the solution r associated to the skew brace B is exactly the same associated to B viewed as semi-brace.

The solution associated to a brace

Theorem (Rump, 2005)

Let $(B,+,\circ)$ be a brace. Then, the function $r: B\times B\to B\times B$ given by

$$r\left(a,b\right)=\left(\ \lambda_{a}\left(b\right),\ \lambda^{-1}_{\lambda_{a}\left(b\right)}\left(a\right)\ \right)$$

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Clearly, we may obtain this basic result as a corollary of the previous theorem by Guarnieri and Vendramin.

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Hence the solution r associated to the brace B is exactly the same associated to B viewed as semi-brace.

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Another classical way to obtain new solutions of the Yang-Baxter equation is by quotient structures and in this sense by ideals. So, we introduce also a suitable concept of *ideal* of a semi-brace in order to obtain new left non-degenerate solutions of the Yang-Baxter equation through *quotient structures* of a given semi-brace.

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Moreover, we focus on a special ideal, the *socle* of a semi-brace, which is a generalization of that already introduced by Rump for braces and then by Guarnieri and Vendramin for skew braces.

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Moreover, we focus on a special ideal, the *socle* of a semi-brace, which is a generalization of that already introduced by Rump for braces and then by Guarnieri and Vendramin for skew braces.

Definition (Catino, Colazzo, P.S., J. Algebra, 2017)

Let B be a semi-brace. We call the set given by

$$\mathrm{Soc}\,(B) = \{a \mid a \in B \quad \lambda_a = \lambda_0, \quad \rho_a = \rho_0\}$$

the **socle** *of the semi-brace B.*

The quotient by the socle and related solution

If B is a semi-brace and consider the quotient of B by its socle $\operatorname{Soc}(B)$, then we may consider the solution associated to the quotient semi-brace $\bar{S} := B/\operatorname{Soc}(B)$

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$$\tilde{r}([a], [b]) = ([\lambda_a(b)], [\rho_b(a)]),$$

for all $a, b \in B$, where we denote by $[a]$ the class of the element a modulo $\operatorname{Soc}(B)$.

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In this way we obtain another left non-degenerate solution other that associated to the semi-brace B.

The quotient by the socle and related solution

If B is a semi-brace and consider the quotient of B by its socle $\operatorname{Soc}(B)$, then we may consider the solution associated to the quotient semi-brace $\bar{S} := B/\operatorname{Soc}(B)$, and so the function $\tilde{r}: \bar{S} \times \bar{S} \to \bar{S} \times \bar{S}$ given by

$$\tilde{r}([a],[b]) = ([\lambda_a(b)],[\rho_b(a)]),$$

for all $a, b \in B$, where we denote by $[a]$ the class of the element a modulo $\operatorname{Soc}(B)$.

In this way we obtain another left non-degenerate solution other that associated to the semi-brace B.

In the classical case of an involutive non-degenerate solution, the solution $\tilde{r}$ is the so-called **retraction** of r, that is widely studied by many authors as we may see, for instance, in
[Rump, 2007], [Cedó, Jespers, Okniński, 2014], [Bachiller, 2015] .

Thank you for your attention!