Hierarchically hyperbolic groups, cubulating groups and RAAGs

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joint work with Federico Berlai

Napoli, 23rd May 2017

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Example

Groups acting on simplicial trees (Bass-Serre theory).

Introduction

- Gromov introduced the notion of hyperbolicity in groups in 1987.

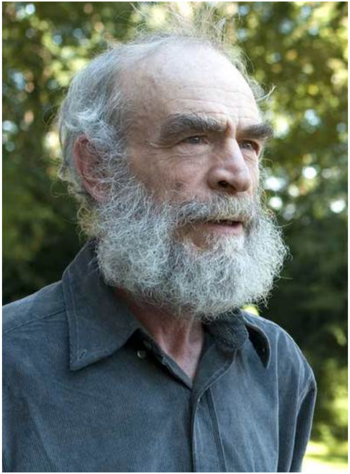

Figura: Mikhail Leonidovich Gromov

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A triangle is δ-thin if it looks like the figure on the right.

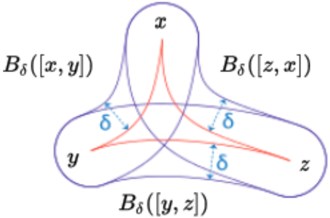

Figura: A δ-thin triangle

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Attempts to generalize hyperbolic groups were made by Bridson; Gilman; Howie and many more:

- Relatively hyperbolic groups;
- $CAT(0)$-groups;
- acylindrically hyperbolic groups.

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Main examples

- Hyperbolic spaces and (direct) product of hyperbolic spaces;
- hyperbolic groups; right-angled Artin groups and mapping class groups (and many more).

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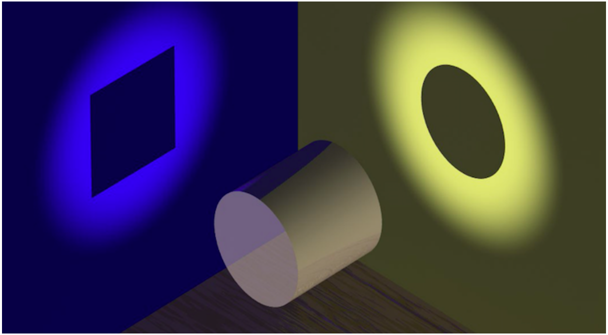

Figura: A space and its projections

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Additional axioms of HHS:

- Bounded geodesic image;
- partial realization;
- uniqueness;
- consistency;
- finite complexity;
- large links.

Distance Formula (J. Behrstock; M.Hagen; A. Sisto, 2015)

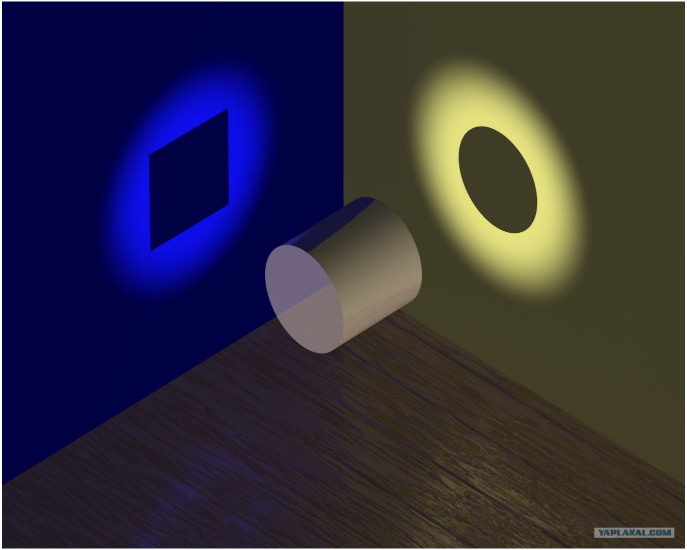
There exists $s_0 >> 0$ such that for every $s > s_0$ there exists K, C such that

$$\sum_{U \in \mathfrak{S}} [d_U(\pi_U(x), \pi_U(y))]_s \asymp_{(K,C)} d(x, y)$$

for every $x, y \in \mathcal{X}$.

Notation

- $A \asymp_{(K,C)} B$ if $K^{-1}A - C \leq B \leq KA + C$;
- $[A]_B = 0$ if $A \leq B$ and $[A]_B = A$ otherwise.



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If G is an HHG, then its Cayley graph is a hierarchically hyperbolic space.

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Theorem

If G_1 and G_2 are HHG then $G_1 \times G_2$ is HHG.

RAAGS are HHG

Definition

Let $\Gamma = (V, E)$ be a finite simplicial graph. The right-angled Artin group (RAAG) wrt Γ is

$$A_\Gamma = \left\langle x_1, \ldots, x_v \mid [x_i, x_j] = 1 \quad \Leftrightarrow \quad \{x_i, x_j\} \in E \right\rangle.$$

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A_Γ is HHG.

Example

$\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$ is HHG.

Build your own HHG!

Definition (Graph of groups)

A graph $\mathcal{T} = (V, E)$ along with groups $\mathcal{G}_v$ for each $v \in V$, $\mathcal{G}_e$ for each $e \in E$ and $\phi_{e^-} : \mathcal{G}_e \to \mathcal{G}_{e^-}$, $\phi_{e^+} : \mathcal{G}_e \to \mathcal{G}_{e^+}$ hieromorphisms is a graph of groups.

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Combination theorem (J. Behrstock; M.Hagen; A. Sisto, 2015)

Let $\mathcal{T}$ be a finite graph of HHG satisfying 'strong' conditions on the edge groups, then the total group

$$G(\mathcal{T}) = \left\langle *_{v \in V} \mathcal{G}_v, E \mid e\phi_{e^+}(g)e^{-1} = \phi_{e^-}(g), e = 1 \text{ for a spanning tree } \right\rangle$$

is a hierarchically hyperbolic group.

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Corollary

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Combination theorem (F.Berlai; R.)

Let $\mathcal{T}$ be a finite graph of hierarchically hyperbolic groups satisfying 'mild' conditions on the edge groups, then the total group $G(\mathcal{T})$ is hierarchically hyperbolic group.

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Let $\mathcal{T}$ be a finite graph of hierarchically hyperbolic groups satisfying 'mild' conditions on the edge groups, then the total group $G(\mathcal{T})$ is hierarchically hyperbolic group.

Remark

This theorem is a generalized version of the Bestvina-Feighn combination theorem for hyperbolic groups.

Graph product

Recall

Given a finite simplicial graph Γ, the RAAG A_Γ associated to it is

$$A_\Gamma = \left\langle x_1, \dots, x_v \mid [x_i, x_j] = e \quad \Leftrightarrow \quad \{x_i, x_j\} \in E \right\rangle$$

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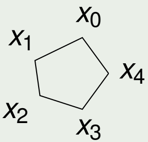
$\Gamma =$ 

$$A_\Gamma = \langle x_0, x_1, x_2, x_3, x_4 \mid [x_i, x_{i+1}] : i \in \mathbb{Z}_5 \rangle$$

Graph product

Definition

Let $\Gamma = (V, E)$ be a graph and $\mathcal{G} = \{G_v\}_{v \in V}$ a collection of groups. The graph product $\Gamma\mathcal{G}$ wrt $\mathcal{G}$ is

$$\Gamma\mathcal{G} = \left\langle *_{v \in V} G_v \mid [G_{v_i}, G_{v_j}] = e \quad \Leftrightarrow \quad \{v_i, v_j\} \in E \right\rangle$$

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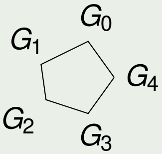

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Remark

For the particular cases of Γ in which

- $E = \varnothing$ (free product);
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it is known that if G_v is HHG for each $v \in V$ then $\Gamma\mathcal{G}$ is HHG.

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Let Γ be a finite graph of HHG, is it true that the graph product $\Gamma(\mathcal{G})$ is a HHG?

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Conjecture

Yes (work in progress).

Final comments

Idea

- Any graph product of groups $\Gamma(G)$ is isomorphic to a total group of a graph of groups $G(\mathcal{T})$.
- Apply combination theorem.

Remark

The positive answer to the question would give a way of building new HHG by 'gluing' them over graph of groups.

Grazie mille!

CS ;)