Groups with a Frobenius group of automorphisms

A bound for their nilpotency class

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Frobenius group

A finite *Frobenius group* FH, with kernel F and complement H, is a semidirect product of a normal subgroup F and a subgroup H such that $C_F(h) = 1$, for all $1 \neq h \in H$;

- the kernel F is nilpotent (Thompson);
- its nilpotentcy class $c(F) \leq h(p)$, where p is the least prime dividing $|H|$ (Higman-Kreknin-Kostrikin);
- all Sylow subgroups of the complement H are cyclic or generalized quaternion;

etc.

Examples: S_3, D_n (for odd n), $\mathbb{F}_q \rtimes \mathbb{F}_q^*$ for every finite field $\mathbb{F}_q$, ...

Frobenius group of automorphisms

Let G be a finite group admitting a Frobenius group of automorphisms $FH \leq Aut(G)$, with kernel F and complement H.

The kernel acts *fixed-point-freely*:

- G is soluble (Belyaev and Hartley);
- if $|F|$ is a prime p, then G is nilpotent of nilpotency class $c(G) \leq h(p)$ (Higman-Kreknin-Kostrikin)

$$\frac{p^2-1}{4} \leq h(p) < \frac{(p-1)^{2^{p-1}-1}}{p-2}$$

Lie ring Methods

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$$C_G(F) = \left\{ g \in G : g^f = g, \ \forall f \in F \right\} = 1.$$

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Lie ring Methods

A bound for the nilpotency class in terms of $C_G(H)$

Theorem

Let G be a finite group admitting a Frobenius group of automorphisms FH, with cyclic kernel F and complement H, such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class c. Then G is nilpotent of class $(c, |H|)$-bounded.

- the bound is exponential in $|H|$ and c;
- Lie ring method+combinatorial tools;
- similar result for a Lie ring under the same hypotheses;
- $C_G(H)$ nilpotent, $C_G(F) = 1$ $\Longrightarrow$ G nilpotent.

Questions:

1) Is the condition on the cyclic Frobenius kernel necessary?
2) Does the nilpotency class of the group really depend on the order of H?

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Two examples

$$L = \frac{\mathbb{Z}}{p^m\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^m\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^m\mathbb{Z}}$$

generated respectively by e_1, e_2, e_3.

$$[e_1, e_2] = pe_3,\ [e_2, e_3] = pe_1,$$
$$[e_3, e_1] = pe_2.$$

$F = \{1, f_1, f_2, f_3\}$ $\qquad$ $H =< h >$, $o(h) = 3$.

$$f_i(e_j) = \begin{cases} e_i, & i = j \\ -e_j, & i \neq j \end{cases} \qquad h(e_j) = e_{j+1(mod\ 3)}$$

$C_L(F) = 0$, $\qquad$ $C_L(H)$ is abelian.

$c(L) = m$

L = 3-dimensional Lie algebra over $\mathbb{K}$ ($ch(\mathbb{K}) \neq 2$), generated by e_1, e_2, e_3.

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A preliminary result for graded Lie rings

Proposition

Let A be a finite abelian group. Let $L = \bigoplus_{a \in A} L_a$ be an A-graded Lie ring which satisfies the following properties:

1. $L_0 = 0$;
2. *there exists a natural number m such that* $|\{N_a\}| = |\{b \in A : [L_b, L_a] \neq 0\}| \leq m$ *for all $a \in A$ (m-condition);*
3. *the number $m < p$, where p is the least prime divisor of $|A|$.*

Then L is nilpotent of $(m, |A|)$-bouded class.

- prove that L is soluble, the nilpotency class will be shown by induction on the derived length;
- solubility proved by induction on the rank of A;

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A partial generalization of the theorem

Theorem

Let G be a finite group admitting a Frobenius group of automorphisms FH, with abelian kernel F and complement H. Suppose that F acts coprimely on G in such a manner that $C_G(F)$ is trivial and $C_G(H)$ is abelian. If $|H|^2$ is less than the least prime divisor of $|F|$, then G is nilpotent of $(|F|,|H|)$-bounded class.

Proof.

1. construction of the *associated Lie ring* L;
2. observe that $L = \bigoplus_{\chi \in \hat{F}} L_\chi$, and $L_0 = C_L(F) = 0$;
3. the Frobenius group $\hat{F}H$ on L, the action of H on the homogeneous components;
4. L satisfies the *m-condition*, where $m = |H|(|H| - 1)$.

...the result follows from the previous proposition. □

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Thank you