# Fusion and pearls

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## Definition

Let $p$ be a prime and let $S$ be a Sylow $p$-subgroup of $G$. The fusion category of $G$ on $S$ is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of $S$ and whose morphism sets are:

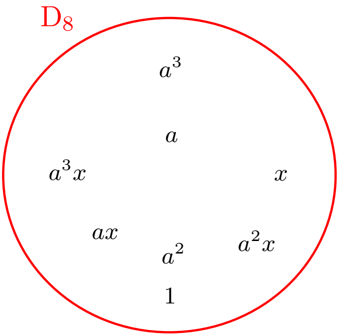
$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \{c_g|_P : P \to Q | g \in G, P^g \leq Q\},$$

for every $P, Q \leq S$.

Pick a $p$-group $S$.

$$S = \mathrm{D}_8$$

$$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$$



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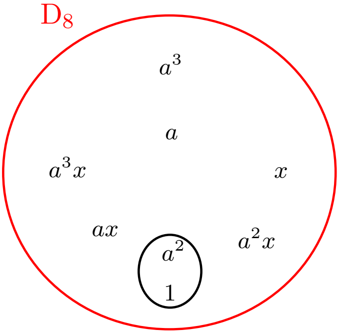

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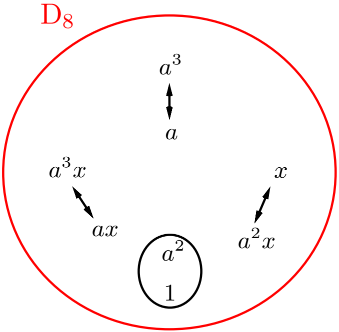


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Consider the conjugation maps by elements $g \in G$, that *fuse* some elements/subgroups of $S$.

*fusion* is determined by $\mathrm{Inn}(\mathrm{D}_8)$

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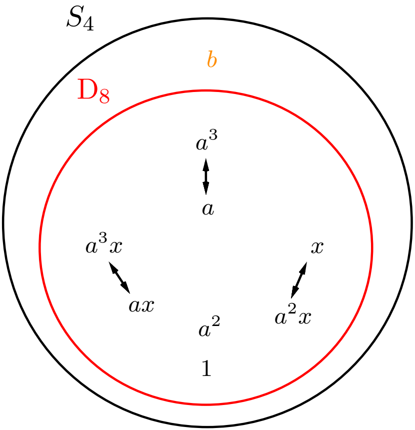


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$a = (1234),\ x = (13).$

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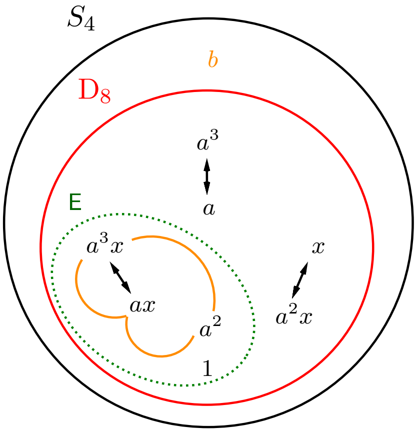


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$a = (1234),\ x = (13).$

$b = (123)$

$(a^2)^b = ((13)(24))^{(123)} = (12)(34) = ax$

*fusion* is determined by

$\mathrm{Inn}(D_8)$ and $\mathrm{Aut}(E) \cong \mathrm{SL}_2(2) \cong S_3$

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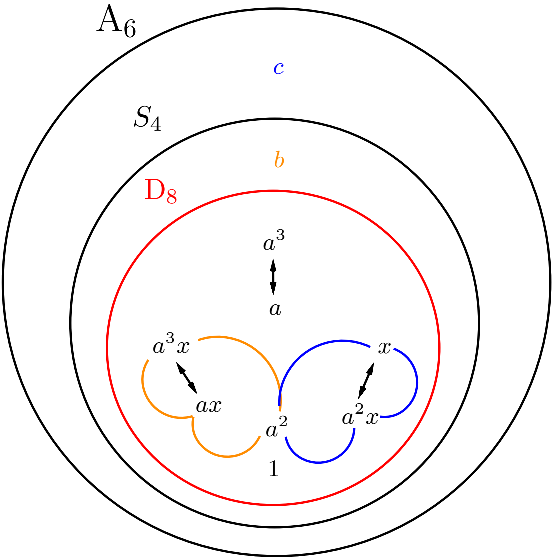


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$G = A_6$

$a = (1234)(56), x = (13)(56).$

$c = (25)(46)$

$(a^2)^c = ((13)(24))^{(25)(46)}$

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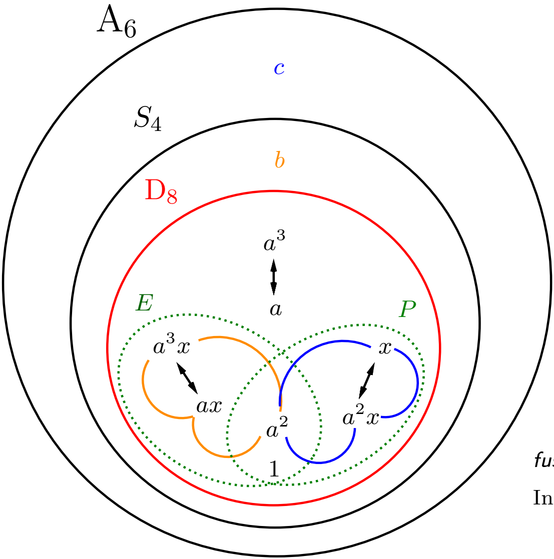


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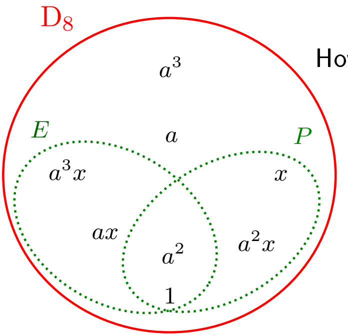


How many ways to fuse elements of $\mathrm{D}_8$?

– $\mathcal{F}_{\mathrm{D}_8}(\mathrm{D}_8)$ (no essential subgroups)

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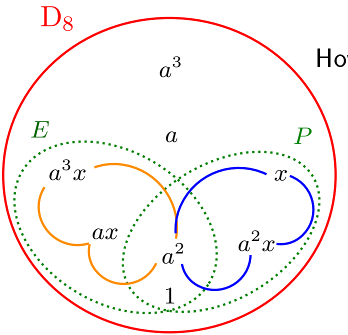


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Let $p$ be a prime and let $S$ be a $p$-group. Choose a collection of morphisms between subgroups of $S$ that "behave" as conjugation maps.

## Definition (Fusion System)

A Fusion System $\mathcal{F}$ on $S$ is a category whose objects are the subgroups of $S$ as and with morphisms $\mathrm{Mor}(\mathcal{F}) \subseteq \bigcup_{P,Q \leq S} \mathrm{Inj}(P, Q)$ such that

1. $\mathrm{Inn}(S) \subseteq \mathrm{Mor}(\mathcal{F})$,
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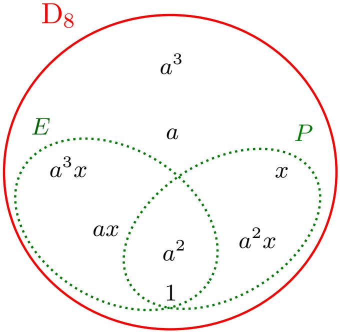
## Alperin-Goldschmidt Fusion Theorem

Let $\mathcal{F}$ be a Saturated Fusion System over the $p$-group $S$. Then $\mathcal{F}$ is completely determined by the automorphism group of certain subgroups of $S$, called Essential Subgroups.

Let $S = \mathrm{D}_8$. $E$ and $P$ are the only subgroups of $S$ that can be essential and $E \cong P \cong \mathrm{C}_2 \times \mathrm{C}_2$.



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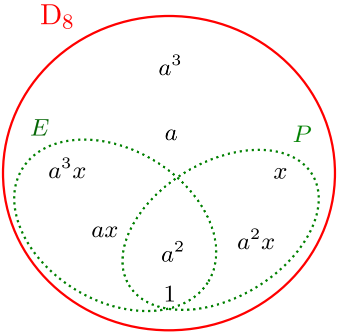


## Question

Suppose that $\mathcal{F}$ is a saturated fusion system on a $p$-group $S$ and there exists an essential subgroup $E$ of $S$ such that $E \cong \mathrm{C}_p \times \mathrm{C}_p$. What can we say about $\mathcal{F}$ and $S$?

# Pearls

Let $\mathcal{F}$ be a saturated fusion system on the $p$-group $S$.

## Definition

A pearl is an essential subgroup $E$ of $S$ that is either elementary abelian of order $p^2$ ($E \cong \mathrm{C}_p \times \mathrm{C}_p$) or non-abelian of order $p^3$ and exponent $p$.

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Note that pearls are the smallest candidates for abelian and non-abelian essential subgroups of a $p$-group.

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**Fact: essential subgroups are self-centralizing**

If $E$ is an essential subgroup of $S$ then

$$\mathrm{C}_S(E) = \{x \in S | ex = xe \text{ for every } e \in E\} \leq E.$$

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**Theorem**

If a $p$-group $S$ contains a pearl then $S$ has maximal nilpotency class.

# $p$-groups having maximal nilpotency class

Let $G$ be a finite group.

Lower central series of $G$: $G_2 := [G, G]$, $G_i := [G_{i-1}, G]$ for every $i \geq 3$.

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$G$ is nilpotent if there exists $t \in \mathbb{N}$ such that $G_{t+1} = 1$ (equivalently $\mathrm{Z}_t(G) = G$), and the smallest $t$ with this property is the nilpotency class of $G$.

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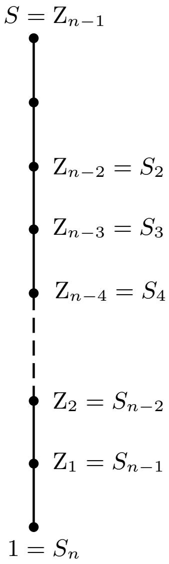
Every finite $p$-group is nilpotent. A $p$-group $S$ of order $p^n$ having nilpotency class $n-1$ is said to have maximal nilpotency class.

# $p$-groups having maximal nilpotency class




Let $S$ be a $p$-group having order $p^n$ and maximal nilpotency class. Then

- $\mathrm{Z}_i(S) = S_{n-i}$ for every $1 \leq i \leq n-2$, $\mathrm{Z}_{n-1}(S) = S$ and $S_n = 1$;

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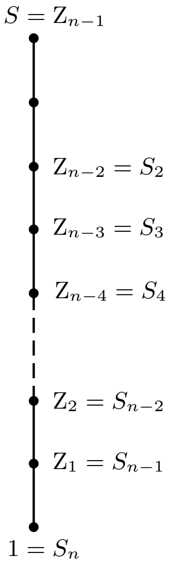


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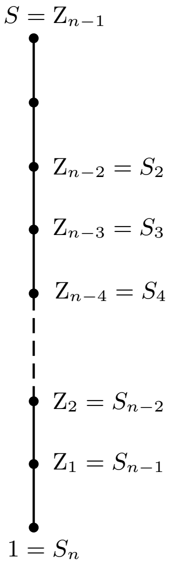
- $Z_i(S) = S_{n-i}$ for every $1 \leq i \leq n-2$, $Z_{n-1}(S) = S$ and $S_n = 1$;
- $|Z_1(S)| = p$, $[Z_i(S) \colon Z_{i-1}(S)] = p$ for every $1 \leq i \leq n-2$ and $S/Z_{n-2} \cong C_p \times C_p$;

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- for every $i \geq 2$, the group $S_i$ is the only normal subgroup of $S$ of order $p^{n-i}$.

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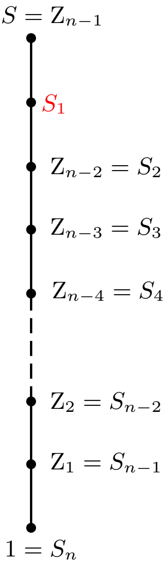


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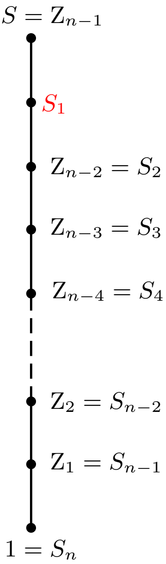


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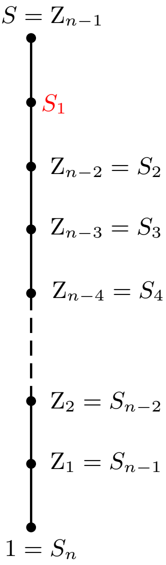


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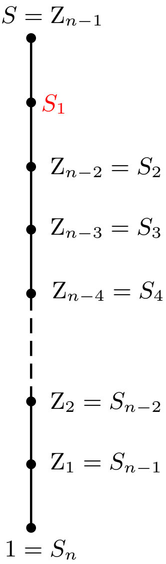


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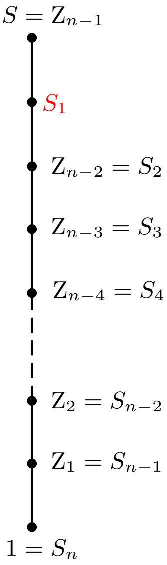


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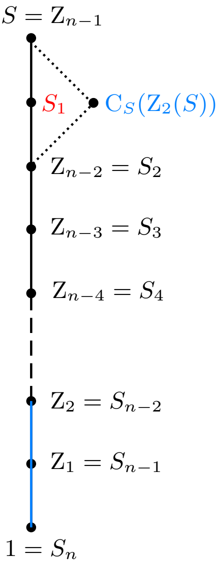


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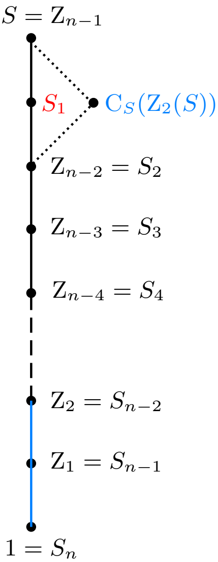


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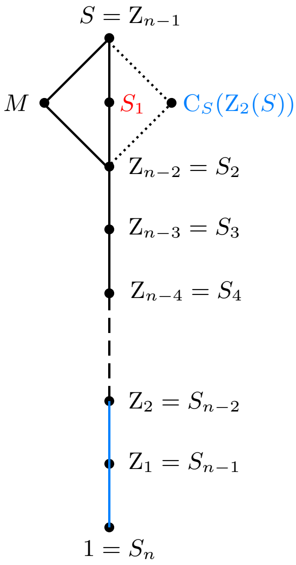


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Every other maximal subgroup of $S$ has maximal nilpotency class.

Let $p$ be a prime and let $\mathcal{F}$ be a saturated fusion system on the $p$-group $S$.

### Definition

A pearl is an essential subgroup $E$ of $S$ that is either elementary abelian of order $p^2$ ($E \cong \mathrm{C}_p \times \mathrm{C}_p$) or non-abelian of order $p^3$ and exponent $p$.

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If a $p$-group $S$ contains a pearl then $S$ has maximal nilpotency class.

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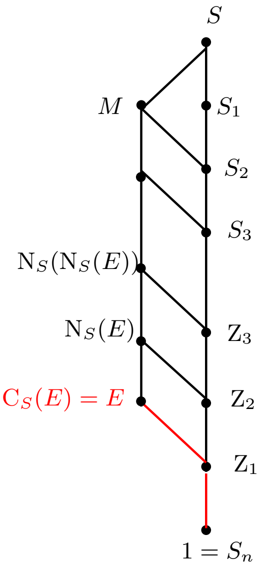
From now on we assume that $p$ is odd. In particular, if $E$ is a pearl then either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ or

$$E \cong p_+^{1+2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} | a, b, c \in \mathrm{GF}(p) \right\} \leq \mathrm{SL}_3(p)$$

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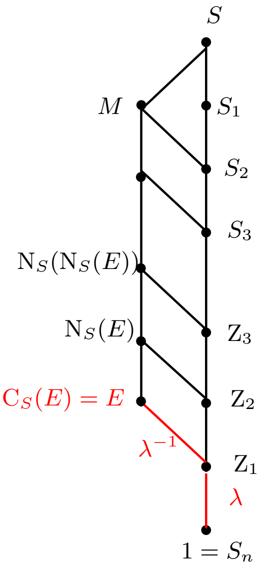
# Structure of a $p$-group containing an abelian pearl




Suppose $E \cong C_p \times C_p$ is an essential subgroup of the $p$-group $S$. Then:

- $C_S(E) = E$;
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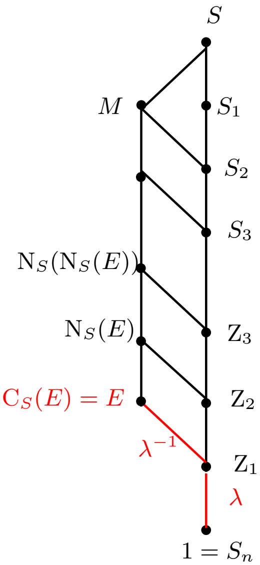


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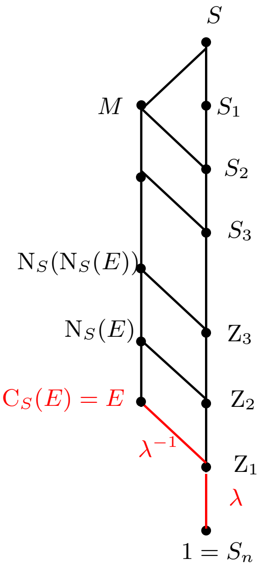


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So if $E = \langle e \rangle \times \langle z \rangle$, with $z \in Z_1$, then

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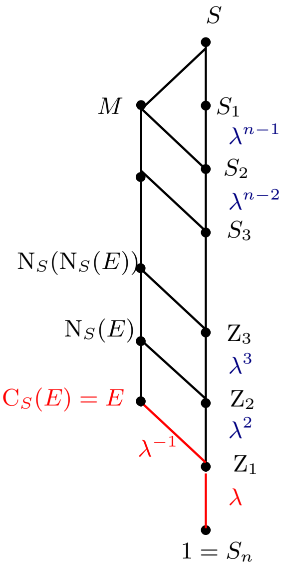


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Note that $\varphi$ acts on every $S_i$ as described by the picture.

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Suppose that $S$ has sectional rank $k$ (every subgroup of $S$ can be generated by at most $k$ elements).

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Suppose that $S$ has sectional rank $k$ (every subgroup of $S$ can be generated by at most $k$ elements). Then $p \geq k \geq 2$ and one of the following holds:

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Suppose that $S$ has sectional rank $k = 3$ (every subgroup of $S$ can be generated by at most $3$ elements). Then $p \geq 3$ and one of the following holds:

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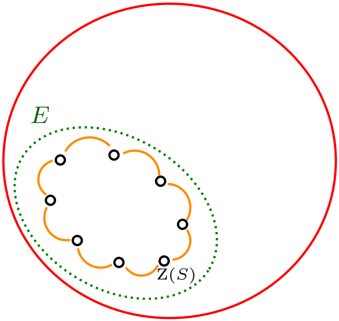

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## Theorem 3

Let $p$ be an odd prime, $p \geq 5$, let $\mathcal{F}$ be a simple fusion system on the $p$-group $S$ and suppose that $S$ has sectional rank $k = 3$.
Then $\mathcal{F}$ contains a pearl (and so $\mathcal{F}$ and $S$ are as described in Theorem 2).

Grazie.