A new family of irretractable left cycle sets

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Definition (Rump, 2005)

Let X be a non-empty set and $\cdot$ a binary operation on X.
The pair $(X, \cdot)$ is said a *left cycle set* if for all $x, y, z \in X$ it holds

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{1}$$

and left multiplication $\sigma_x : X \longrightarrow X,\ y \longmapsto x \cdot y$ is bijective, for every $x \in X$.
We call $(X, \cdot)$ *non-degenerate* if the squaring map

$$\mathfrak{q} : X \longrightarrow X,\ x \longmapsto x \cdot x$$

is bijective and it is said *square-free* if $\mathfrak{q} = id_X$.

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Example

If X is a non-empty set, then $(X, \cdot)$, where $m \cdot n := n$ for all $m, n \in X$, is a non-degenerate left cycle set.

Example

The left cycle set $(\mathbb{Z}, \cdot)$, where $m \cdot n := n - min\{m, 0\}$, is an example of degenerate left cycle set.

Theorem (Rump, 2005)

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A *set-theoretic solution of the Yang-Baxter equation* on a set X is the pair (X, r), where the map $r : X \times X \to X \times X$ is such that

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where $r_1 := r \times id_X$ and $r_2 := id_X \times r$.

Problem (Drinfield, 1992)

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A set-theoretic solution of the Yang-Baxter equation
$r : X \times X \to X \times X, \quad (x, y) \to (\lambda_x(y), \rho_y(x))$ is called:

1) *involutive* if $r^2 = id_{X \times X}$;

2) *non-degenerate* if $\lambda_x, \rho_x \in Sym(X)$ for every $x \in X$;

3) *square-free* if $r(x, x) = (x, x)$ for every $x \in X$.

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Theorem (Rump,2004)

Let $(X,\cdot)$ be a non-degenerate left cycle set and $r: X\times X \longrightarrow X\times X$ given by $r(x,y) := (\lambda_x(y), \rho_y(x))$, where

$$\lambda_x(y) := \sigma_x^{-1}(y) \text{ and } \rho_y(x) := \sigma_x^{-1}(y)\cdot x.$$

Then (X,r) is a solution of the Yang-Baxter equation and it is called **associated solution**.
Conversely, if (X,r) is a solution of the Yang-Baxter equation then the pair $(X,\cdot)$ is a non-degenerate left cycle set, where the operation is given by $x\cdot y := \lambda_x^{-1}(y)$ for all $x, y \in X$.
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Let X be a non-degenerate left cycle set and $\sim$ the relation on X given by

$$x \sim y :\Leftrightarrow \sigma_x = \sigma_y.$$

Then $\sim$ is a congruence of X and $X/\sim$ is a left cycle set whenever X is non-degenerate.

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A left cycle set $(X, \cdot)$ is said *irretractable* if $X = X/\sim$, otherwise X is called *retractable*.

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Let (X, r) be a solution of the Yang-Baxter equation and $\sim'$ the relation on X given by

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Proposition

Let (X, r) be a non-degenerate involutive solution of the Yang-Baxter equation and $(X, \cdot)$ the associated left cycle set. Then $\sim = \sim'$.

So a solution is called irretractable (resp. retractable) if and only if the associated left cycle set is irretractable (resp. retractable).

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Example

Let X be a non-empty set and $\alpha \in Sym(X)$. Let $\cdot$ be the binary operation on X given by

$$x \cdot y := \alpha(y)$$

for all $x, y \in X$.
Then $(X, \cdot)$ is a retractable left cycle set: indeed, $\sigma_x = \alpha$ for every $x \in X$.

Example

Let $X := \{1, 2, 3, 4\}$ and $\cdot$ be the operation on X given by

$$i \cdot j := \sigma_i(j)$$

for all $i, j \in X$, where $\sigma_i \in Sym(X)$ for all $i \in X$ and they are given by:

$$\sigma_1 := (34) \quad \sigma_2 := (1423) \quad \sigma_3 := (1324) \quad \sigma_4 := (12)$$

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Every square-free left cycle set X such that $2 \leq |X| < \infty$ is retractable.

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In 2015 Vendramin found a counterexample to the Gateva-Ivanova Strong Conjecture.

Example (Vendramin, 2015)

Let $X := \{1, ..., 8\}$ and put

$$\sigma_1 := (57); \quad \sigma_2 := (68); \quad \sigma_3 := (26)(48)(57); \quad \sigma_4 := (15)(37)(68);$$

$$\sigma_5 := (13); \quad \sigma_6 := (24); \quad \sigma_7 := (13)(26)(48); \quad \sigma_8 := (15)(24)(37);$$

then $(X, \cdot)$ is a square-free irretractable left cycle set.

This is the square-free irretractable left cycle set of minimal cardinality.

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The multiplication table of the previous example is the following:

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Theorem (Bachiller, Cedó, Jespers and Okniński, 2016)

Let A and B be non-trivial abelian groups and let I be a set with $|I| > 1$. Let $\varphi_1 : A \longrightarrow B$ be a function such that $\varphi_1(-a) = \varphi_1(a)$ for every $a \in A$ and let $\varphi_2 : B \longrightarrow A$ be a homomorphism.
On $X(A, B, I) := A \times B \times I$ we define the following operation

$$(a, b, i)\cdot(c, d, j) := \begin{cases} (c, d - \varphi_1(a - c), j), & \text{if } i = j \\ (c - \varphi_2(b), d, j), & \text{if } i \neq j \end{cases}$$

for all $a, c \in A$, $b, d \in B$ and $i, j \in I$.
Then $(X(A, B, I), \cdot)$ is a non-degenerate left cycle set and it is irretractable and square-free if $\varphi_1^{-1}(\{0\}) = \{0\}$ and φ_2 is injective.

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A larger family of irretractable left cycle set is obtained by M. C., Francesco Catino and Giuseppina Pinto.

Let A, B be non trivial sets and I a left cycle set, $\beta : A \times A \times I \longrightarrow Sym(B)$ and $\gamma : B \longrightarrow Sym(A)$. Put $\beta_{(a,b,i)} := \beta(a, b, i)$, $\gamma_a := \gamma(a)$ and let $\cdot$ be the operation on $A \times B \times I$ given by

$$(a, b, i)\cdot (c, d, j) := \begin{cases} (c,\ \beta_{(a,c,i)}(d),\ i\cdot j), & \text{if } i = j \\ (\gamma_b(c),\ d,\ i\cdot j), & \text{if } i \neq j \end{cases}. \tag{2}$$

Then we write $X(A, B, I, \beta, \gamma)$ to indicate $(A \times B \times I, \cdot)$

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Theorem (M. C., F. Catino, G. Pinto, 2017)

Assume that for $X(A, B, I, \beta, \gamma)$

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2) $\beta_{(a,c,i)} = \beta_{(\gamma_b(a), \gamma_b(c), j \cdot i)}$,
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hold for all $a \in A$, $b, c \in B$, $i, j \in I$, $i \neq j$. *Then* $X(A, I, \beta, \gamma)$ *is a non-degenerate left cycle set.*

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4) $\beta_{(a,c,i \cdot i)} \beta_{(b,c,i)} = \beta_{(b,c,i \cdot i)} \beta_{(a,c,i)}$.

hold for all $a \in A$, $b, c \in B$, $i, j \in I$, $i \neq j$. Then $X(A, I, \beta, \gamma)$ is a non-degenerate left cycle set.

Theorem (M. C., F. Catino, G. Pinto, 2017)

Assume that for $X(A, B, I, \beta, \gamma)$

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Observation

Let I be the left cycle set given by $x \cdot y = y$ for all $x, y \in I$, $\varphi_1 : A \longrightarrow B$ a function such that $\varphi_1(-a) = \varphi_1(a)$ for every $a \in A$ and let $\varphi_2 : B \longrightarrow A$ be an homomorphism. Put

$$\beta_{(a,c,i)} := t_{-\varphi_1(a-c)} \quad \text{and} \quad \gamma_b := t_{-\varphi_2(b)}$$

for all $a, c \in A$, $b \in B$ and $i \in I$, where t_v is the translation by v. Then $X(A, B, I, \beta, \gamma)$ is the non-degenerate left cycle set $X(A, B, I)$ obtained by Bachiller, Cedó, Jespers and Okniński.

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Proposition (M.C., F. Catino, G. Pinto, 2017)

Let A, B, I, β, γ defined as previous Theorem and assume

1) $A \times A \times \{i\} \cap \beta^{-1}(\beta_{(a,a,i)}) \subseteq \{(k, k, i) | k \in A\}$;
2) γ *is injective;*

for all $i \in I$ and $a \in A$.
Then the left non-degenerate cycle set $X(A, B, I, \beta, \gamma)$ is irretractable.

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The following left cycle set is an example of irretrattabile square-free left cycle set different from those obtained last year by Bachiller, Cedó, Jespers e Okniński.

Example

Let $I := \{1, 2\}$, $A = B := \mathbb{Z}/4\mathbb{Z}$ and $\alpha := (1 \quad 2 \quad 3 \quad 4)$. Put

$$\beta_{(a,a,i)} := id_A \qquad \beta_{(a,b,1)} := \alpha$$

$$\beta_{(a,b,2)} := \alpha^2 \qquad \gamma_a := t_{-a-1}$$

for all $i \in I$ and $a \in A$, $b \in B$, $a \neq b$ where $t_a : A \to A$, $t_a(x) := x + a$ for all $a \in A$.
Then $X(A, B, I, \beta, \gamma)$ is an irretractable square-free left cycle set of cardinality 32.

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Thanks!