On the Zassenhaus conjecture for direct products

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Integral group rings

G a finite group.

The Integral group ring

$$\mathbb{Z}G = \left\{ \sum_{g \in G} u_g g \ : \ u_g \in \mathbb{Z} \ \text{ for every } g \in G \right\}$$

$$U(\mathbb{Z}G) = \{\text{Units of } \mathbb{Z}G\}$$

The elements $\pm g$ with $g \in G$ are called **trivial units**.

G. Higman. *The units of group-rings.* Pro. London Math. Soc. (2), 46:231-248, 1940.

Notation

The Augmentation map

$$\begin{array}{rccc} \varepsilon : & \mathbb{Z}G & \to & \mathbb{Z} \\ & \sum_{g\in G} u_g g & \to & \sum_{g\in G} u_g \end{array}$$

Units with augmentation one

- $V(\mathbb{Z}G) = \{u \in U(\mathbb{Z}G) \ : \ \varepsilon(u) = 1\}$.
- $U(\mathbb{Z}G) = \pm V(\mathbb{Z}G)$.

Torsion units with augmentation one

General problem

How are the torsion elements of $V(\mathbb{Z}G)$?

- (Berman-Higman) If G is an abelian finite group then every torsion element of $V(\mathbb{Z}G)$ is an element of G.
- Obvious torsion units: Conjugates of elements of G.

Example

- There is a torsion unit in $V(\mathbb{Z}S_3)$ which is not conjugate in $\mathbb{Z}S_3$ to any element of S_3.
- But it is conjugate in $U(\mathbb{Q}S_3)$.

The Zassenhaus Conjecture

The Zassenhaus Conjecture (1974)

Every torsion unit of $V(\mathbb{Z}G)$ is conjugate in $U(\mathbb{Q}G)$ to an element of G.

The Zassenhaus Conjecture has been proved for:

- Nilpotent groups. (Weiss 1991)
- Metacyclics groups. (Hertweck 2008)
- Cyclic-by-abelian groups. (Caicedo, Margolis and del Río 2013)
- A_6. (Hertweck 2006)
- $\mathrm{PSL}(2, p)$ for p a Fermat or Mersenne prime. (Margolis, del Río and S. 2016)
- Groups till order 143. (Bächle, Herman, Konovalov, Margolis and Singh 2016)

Notation

Denote by g^G the conjugacy class of g in G.

Partial augmentations in $\mathbb{Z}G$

- $u = \sum_{g \in G} u_g g \in \mathbb{Z}G$ and $h \in G$.
- $\varepsilon_h(u) = \sum_{g \in h^G} u_g$ the partial augmentation of u with respect to h.

A result to deal with the Zassenhaus conjecture

Marciniak, Ritter, Sehgal and Weiss.

G a finite group. The following conditions are equivalent:

1. The Zassenhaus Conjecture holds for G.
2. For every torsion element $u \in V(\mathbb{Z}G)$, every $d \mid |u|$ and every $g \in G$ we have $\varepsilon_g(u^d) \geq 0$.

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General problem for direct products

Problem

G and H finite groups satisfying the Zassenhaus conjecture.
Does the Zassenhaus conjecture hold for $G \times H$?

Known results

Höfert 2004

G finite group for which the Zassenhaus conjecture holds.
Then it also holds for $G \times C_2$.

Hertweck 2008

G finite group for which the Zassenhaus conjecture holds.
H nilpotent group with $\gcd(|G|, |H|) = 1$.
Then the Zassenhaus conjecture holds for $G \times H$.

Motivation

Proposition

G finite group for which the Zassenhaus conjecture holds.
A abelian finite group.
If all the conjugacy classes of G has at most size 3 then the Zassenhaus conjecture holds for $G \times A$.

Corollary

The Zassenhaus conjecture holds for $S_3 \times A$, where A is any abelian finite group.

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Camina groups

Denote by G' the derived subgroup of the finite group G.

Definitions (1978)

- G is called a Camina group if $gG' = g^G$ for every $g \in G \setminus G'$.
- For a positive integer n, a Camina group G is called an n-Camina group if G' is the union of n G-conjugacy classes.

The Zassenhaus conjecture holds for Camina groups.

Examples

- 1-Camina groups are precisely the abelian finite groups.
- S_3, A_4 and D_8 are 2-Camina groups.
- $C_2^4 \rtimes C_3$ is a 6-Camina group.

Classifying Camina groups

The general classification by Dark and Scoppola 1996

A finite non-abelian group is a Camina group if and only if it is a Camina p-group or a Frobenius group whose complement is either cyclic or Q_8.

Cangelmi and Muktibodh 2010

- 2-Camina group if and only if it is either an extraspecial 2-group or isomorphic to $C_p^r \rtimes C_{p^r-1}$ for a prime p.
- 3-Camina group if and only if it is either an extraspecial 3-group, or isomorphic to $C_p^r \rtimes C_{\frac{p^r-1}{2}}$ for a prime p, or isomorphic to Q_8.

The result

Theorem (Bächle, Kimmerle and S. 2017)

G Camina group. A abelian finite group.
Then the Zassenhaus conjecture holds for $G \times A$.

Thanks for your attention.