Weakly power automorphisms of groups

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An automorphism α of a group G is a power automorphism if it maps every subgroup of G onto itself.

The set of all power automorphisms of G is usually denoted by $PAut(G)$.

The behavior of power automorphisms was described by C.D.H. Cooper (1968).

- Any power automorphism of a group G is central, i.e. it acts trivially on the factor group $G/Z(G)$.
- $PAut(G)$ is a normal abelian subgroup of the full automorphism group $Aut(G)$.

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We shall say that an automorphism α of a group G is a *weakly power automorphism* if $X^\alpha = X$ for every non-periodic subgroup X of G.

The set of all weakly power automorphisms of G will be denoted by $WAut(G)$.

- $WAut(G) \geqslant PAut(G)$.
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If G is any group, the subgroup generated by all elements of infinite order of G will be denoted by $W(G)$.

G is said to be *weak* if $W(G) = G$.

- $Z \wr Z_2$ is a weak groups in which the set of all elements of finite order is not a subgroup
- If G is a non-abelian weak group, then $PAut(G) = \{1\}$
- If G is a non-periodic abelian group, then $|PAut(G)| = 2$

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Theorem 1 (MDF, F. de Giovanni, C. Musella, Y.P. Sysak)

Let G be a weak group.
Then $WAut(G) = PAut(G)$.

In any group G, there exists the largest normal locally nilpotent subgroups, called the *Hirsch-Plotkin radical* of G.

Theorem 2 (MDF, F. de Giovanni, C. Musella, Y.P. Sysak)

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Theorem 2 (MDF, F. de Giovanni, C. Musella, Y.P. Sysak)

Let G be a group whose Hirsch-Plotkin radical is not periodic. Then $WAut(G) = PAut(G)$.

Theorem 3 (MDF, F. de Giovanni, C. Musella, Y.P. Sysak)

Let G be a soluble non-periodic group.
If G has derived length at most 3, then $WAut(G) = PAut(G)$.

Proof

Assume for a contradiction that there exists a weakly power automorphism α of G which is not a power automorphism.

Let x be an element (of finite order) of G such that the subgroup $X = \langle x \rangle$ is not fixed by α.

G is not weak $\Rightarrow$ G/G' is periodic $\Rightarrow$ G' is not periodic.

Let E be a finitely generated non-periodic subgroup of G such that $X \leqslant E \leqslant XG'$ $\Rightarrow$

$\Rightarrow$ $E = X(E \cap G')$ $\Rightarrow$ $E \cap G'$ is finitely generated $\Rightarrow$

$\Rightarrow$ $E \cap G'$ is residually finite (since it is metabelian and finitely generated).

Let K be subgroup of finite index of $E \cap G'$ such that $X \cap K = \{1\}$, and

let $(K_i)_{i \in I}$ be a collection of E-invariant subgroups of finite index of K such that

$$\bigcap_{i \in I} K_i = \{1\}.$$

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$$\bigcap_{i \in I} XK_i = X.$$

Each K_i has finite index in E and hence is not periodic $\Rightarrow$
$\Rightarrow$ α maps $(XK_i)^\alpha = XK_i \ \forall i \in I$ $\Rightarrow$
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Let G be a non-periodic group.Then the following conditions hold:

(1) If α is any weakly power automorphism of G, α acts trivially on the factor group $G/Z(W(G))$. Moreover, if α is not a power automorphism, then it acts trivially also on $W(G)$.

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Let R be the group $F/[F'', F]$ where F is a free non-abelian group with two generators. Then

- R is generated by a subset $\{a, b\}$ of order two
- $Z(R) = R''$ is a free abelian group of countable rank (J.N. Ridley, 1970)
- R'' has a basis consisting of all commutators $[[a,b]^a, [a,b]]$, $[[a,b]^b, [a,b]]$ $[[a,b]^{ab}, [a,b]]$ and $[[a,b]^{a^{r-j+l}}, [a,b]^{b^{-i+k-r}}]$ for all $r \geqslant 1,\ 0 \leqslant k,\ i \leqslant r-1,\ 1 \leqslant l,\ j \leqslant r$
- Put $z = [[a,b]^a, [a,b]]$ and let N be the subgroup generated by all these commutators with the exclusion of z.
- Then $R'' = N \times \langle [[a,b]^a, [a,b]] \rangle$

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Put $K = R/((R')^3N)$. Then

- $K = \langle \bar{a}, \bar{b} \rangle$, where $\bar{a} = a((R')^3N)$, $\bar{b} = b((R')^3N)$
- K/K' is free abelian of rank 2
- K' is an infinite group of exponent 3, and $K'' = \langle \bar{z} \rangle$ where $\bar{z} = [[\bar{a}, \bar{b}]^{\bar{a}}, [\bar{a}, \bar{b}]]$
- The positions $\bar{a}^x = \bar{b}$ and $\bar{b}^x = \bar{a}^{-1}\bar{b}^{-1}$ give rise to an automorphism x of K which has order 3

Put $G = \langle x \rangle \ltimes K$.

In particular, G is soluble of derived length 4.

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In 1934 R. Baer defined the norm $N(G)$ of a group G as the intersection of all normalizers of subgroups of G. Clearly an element g of G belongs to $N(G)$ if and only if it determines, by conjugation, a power automorphism of G.

If $\mathfrak{X}$ is a group class, the $\mathfrak{X}$-*norm* of a group G is defined as the intersection of all normalizers of subgroups of G, which are not in $\mathfrak{X}$.

In recent years, several authors have investigated the behavior of the $\mathfrak{X}$-norm of a group for different choices of the class $\mathfrak{X}$ (for an account of this subject see the recent survey paper:
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If $\mathfrak{X}$ is the class of all periodic groups, we shall denote the $\mathfrak{X}$-norm of G by $WN(G)$.

$WN(G)$ will be called the *weak norm* of G.

- An element g of G belongs to $WN(G)$ if and only if it determines, by conjugation, a weakly power automorphism of G.
- In particular, $WN(G) = N(G)$ whenever $WAut(G) = PAut(G)$.

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Let G and α be the group and the automorphism considered in the example seen above.

Let G^* be the group $\langle\alpha\rangle \ltimes G$.

Then $WN(G^*) = \langle\alpha\rangle \times Z(G)$, while $N(G^*) = Z(G)$.

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Theorem 5 (MDF, F. de Giovanni, C. Musella, Y.P. Sysak)

Let G be a group whose weak norm is non-periodic.
Then $WN(G) = N(G)$.

Theorem 6 (MDF, F. de Giovanni, C. Musella, Y.P. Sysak)

Let G be a non-periodic group.
Then the weak norm $WN(G)$ of G is contained in $Z(W(G))$.

Thank you!