NORMALLY ζ-REVERSIBLE PROFINITE GROUPS

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UNIVERSITÀ
DEGLI STUDI
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These series deal with two important problems arised in the last century:

- the subgroup growth of a profinite group;
- the probability of selecting a generating set randomly choosing a given number of elements of the group.

Let G be a profinite group, let $\{a_n(G)\}$ be the sequence counting the number of open subgroups of index n in G. Assume that G has the property that $a_n(G)$ is finite for any $n \in \mathbb{N}$.

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Then we can consider the Dirichlet series associated to the sequence $\{a_n(G)\}$:

Definition

$$\zeta_G(s) := \sum_n \frac{a_n(G)}{n^s}.$$

$\zeta_G(s)$ is called is the subgroup zeta function associated to G.

Let $\mathcal{L}$ be the lattice of all subgroups of G: then we can define a Möbius function on it in the following way.

Definition

$$\mu(G, G) = 1;$$

$$\mu(H, G) = -\sum_{H<K\leq G} \mu(K, G) \text{ for } H < G.$$

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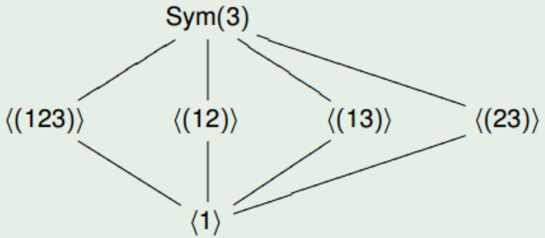
Using $\mu(\cdot, G)$ we are able to define new coefficients associated to G.

Definition

$$b_n(G) = \sum_{|G:H|=n} \mu(H, G)$$

Example

$G = \mathrm{Sym}(3)$



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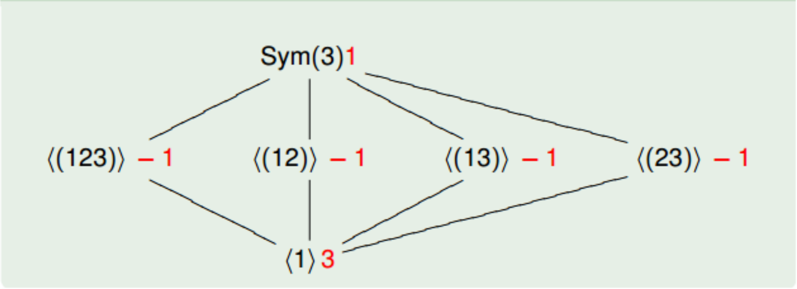

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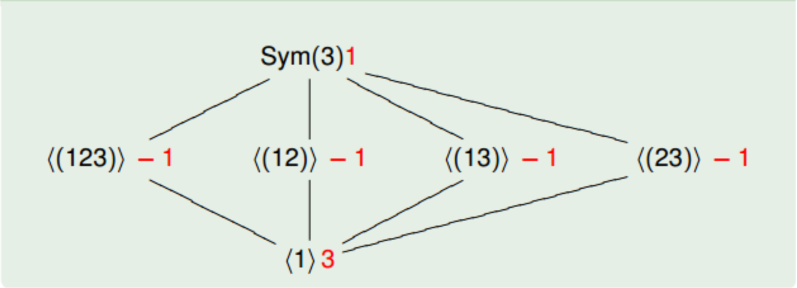

$$b_1(G) = \mu(G, G) = 1;$$
$$b_2(G) = \mu(\langle 1,2,3\rangle, G) = -1;$$
$$b_3(G) = \mu(\langle 1,2\rangle, G) + \mu(\langle 1,3\rangle, G) + \mu(\langle 2,3\rangle, G) = -3;$$
$$b_6(G) = \mu(1, G) = 3.$$

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is the inverse of the probabilistic zeta function associated to G.

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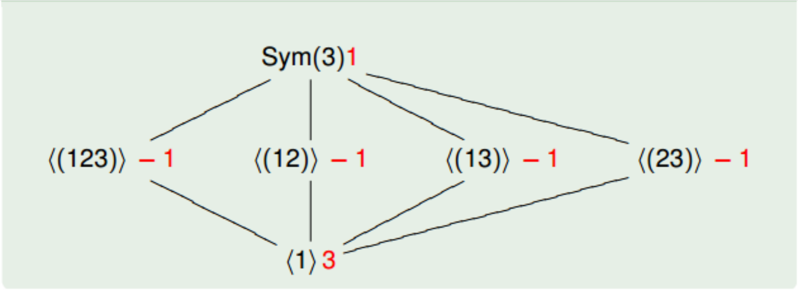

$$\zeta_G(s) = 1 + \frac{1}{2^s} + \frac{3}{3^s} + \frac{1}{6^s}$$

$$p_G(s) = 1 - \frac{1}{2^s} - \frac{3}{3^s} + \frac{3}{6^s}$$

If G is finite, then $p_G(t)$, for a non-negative integer t, has an important probabilistic meaning.

Proposition (Hall, 1936)

Let G be a finite group and $t \in \mathbb{N}$: then $p_G(t)$ is the probability that t randomly chosen elements of G generate G.

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Proposition (Hall, 1936)

Let G be a finite group and $t \in \mathbb{N}$: then $p_G(t)$ is the probability that t randomly chosen elements of G generate G.

Using this probabilistic meaning, it is quite easy to compute $p_G(t)$ for some classes of finite groups.

Proposition

Let G be a p-group and $d = d(G)$. Then

$$p_G(t) = \prod_{j=0}^{d-1} \left(1 - p^{j-t}\right).$$

Problem

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Let G be a finitely generated profinite group with a normalized Haar measure ν and $\Phi_G(t)$ the set of all ordered t-uples generating G. It can be proved that $\Phi_G(t)$ is measurable, thus we can define

$$\text{Prob}_G(t) := \nu(\Phi_G(t)).$$

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if this sum is absolutely convergent, then the Dirichlet series $p_G(s)$ can be obtained from this infinite sum, grouping together all terms with the same denominator, so in particular $p_G(s)$ converges in some right half-plane and it can be proved that $p_G(t) = \text{Prob}_G(t)$, when $t \in \mathbb{N}$ is large enough.

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Example

Consider the profinite complexion of the infinite cyclic group $\mathbb{Z}$. Then

$$\zeta_{\hat{\mathbb{Z}}}(s) = \sum_n \frac{1}{n^s} = \zeta(s)$$

$$p_{\hat{\mathbb{Z}}}(s) = \sum_n \frac{\mu(n)}{n^s}$$

and it is easy to prove that

$$\zeta_{\hat{\mathbb{Z}}}(s) p_{\hat{\mathbb{Z}}}(s) = 1.$$

It is natural to ask if this behaviour is common to a wider class of groups. To this extent, in 2014 Damian and Lucchini introduced the following definition.

Definition

A finitely generated profinite group G is ζ-reversible if $\zeta_G(s)p_G(s) = 1$.

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$\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$ for any prime p are ζ-reversible.

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The interest in ζ-reversible groups is motivated by computational difficulties in finding the coefficients $a_n(G)$ for most groups. In fact, while the series $p_G(s)$ apparently has a more complicated definition than $\zeta_G(s)$, several progresses have been achieved in the last decades in its computation.

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In response to this question, in 2007 Detomi and Lucchini introduced some generalizations of the subgroup and the probabilistic zeta functions.
Let $\mathcal{L}^{\triangleleft}$ be the lattice of normal subgroups of G. Then it is possible to define:

- a Möbius function $\mu^{\triangleleft}(\cdot, G)$;

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- a Möbius function $\mu^{\triangleleft}(\cdot, G)$;
- coefficients $a_n^{\triangleleft}(G)$ (the number of open normal subgroups of G of index n);
- coefficients $b_n^{\triangleleft}(G) = \sum_{H \trianglelefteq G, |G:H|=n} \mu^{\triangleleft}(H, G)$.

Provided that $a_n^{\triangleleft}(G)$ is finite for any n, we can define the two Dirichlet series

$$\zeta_G^{\triangleleft}(s) = \sum_n \frac{a_n^{\triangleleft}(G)}{n^s}$$

and

$$p_G^{\triangleleft}(s) = \sum_n \frac{b_n^{\triangleleft}(G)}{n^s}.$$

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Again, $p_G^{\triangleleft}(t)$ has a probabilistic meaning for $t \in \mathbb{N}$.

Proposition

Let G be a finite group, then $p_G^{\triangleleft}(t)$ is the probability that t randomly chosen elements of G normally generate G.

Proposition (Detomi and Lucchini, 2007)

Let G be a finite group, and $\mathcal{N}(G)$ be the intersection of all maximal normal subgroups of G; let

$$G/\mathcal{N}(G) \cong \prod_{i=1}^{m} S_i^{n_i}$$

where S_i are non-isomorphic simple groups. Then

$$p_G^{\triangleleft}(t) = \prod_{i=1}^{m} p_{S_i^{n_i}}^{\triangleleft}(t).$$

Moreover, for a simple group S,

$$p_{S^n}^{\triangleleft}(t) = \begin{cases} (1 - 1/|S|^t)^n & \text{if S is not abelian;} \\ \prod_{j=1}^{n}(1 - p^{j-1}/p^t) & \text{if S is abelian of order p.} \end{cases}$$

Problem

What happens in the profinite case?

For any profinite group G, we can define $\mathrm{Prob}_G^{\triangleleft}(t) = \nu(\Phi_G^{\triangleleft}(t))$. Again, we need the absolute convergence of the infinite sum

$$\sum_{H \trianglelefteq_O G} \frac{\mu^{\triangleleft}(H, G)}{|G : H|^s}. \tag{1}$$

Theorem (C. and Lucchini, 2016)

Let G be a profinite group such that $a_n^{\triangleleft}(G)$ is finite for every $n \in \mathbb{N}$. Then the following are equivalent:

(i) *the infinite sum* (1) *absolutely converges in some right complex half plane;*

(ii) $\mu^{\triangleleft}(H, G)$ *and* $c_n^{\triangleleft}(G)$ *(the number of open normal subgroups of index n such that $\mu^{\triangleleft}(H, G) \neq 0$) are polynomially bounded in $|G : H|$ and in n respectively;*

(iii) *G is PFNG (i.e., $\mathrm{Prob}_G^{\triangleleft}(t) > 0$ for some $t \in \mathbb{N}$);*

(iv) *G has polynomial maximal normal subgroups growth;*

(v) *$G/\mathcal{N}(G)$ is finitely generated.*

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(v) *$G/\mathcal{N}(G)$ is finitely generated.*

Moreover, if (one of) the previous conditions hold(s), then

$$\mathrm{Prob}_G^{\triangleleft}(t) = p_G^{\triangleleft}(t)$$

for $t \in \mathbb{N}$ big enough.

Definition

A profinite group G is normally ζ-reversible if $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$.

Conjecture

Normally ζ-reversible profinite groups are pronilpotent.

An evidence for our conjecture is given by the following results.

Proposition (Detomi and Lucchini, 2007)

The series $p_G^{\triangleleft}(s)$ is multiplicative if and only if there is no open normal subgroup N in G such that G/N is a nonabelian simple group.

Proposition (Puchta, 2001)

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Proposition (Puchta, 2001)

If $\zeta_G^{\triangleleft}(s)$ is multiplicative, then G is pronilpotent.

Proposition

Let G be a normally ζ-reversible profinite group with the property that there is no $N \triangleleft_O G$ such that G/N is a nonabelian simple group: then G is pronilpotent. In particular, any prosoluble normally ζ-reversible profinite group is pronilpotent.

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We will prove that the conjecture holds in these two cases: G is perfect, all the nonabelian composition factors of G are alternating groups.

Assume that G is a normally ζ-reversible group that is a counterexample to our conjecture.

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Let $m = m_1 < m_2 < \cdots$ be the orders of all nonabelian simple groups which are continuous epimorphic images of G; let t_i (with $t = t_1$) be the number of open normal subgroups N of G such that G/N is a nonabelian simple group of order m_i.

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Then we get from a direct computation that

$$a^{\triangleleft}_{m^2}(G) = \gamma_{m^2}(G) + \sum_{m_i r = m^2} t_i \gamma_r(G) + \binom{t}{2} + t, \tag{2}$$

where $\gamma_i(G)$ is the number of normal subgroups of G of index i with a nilpotent quotient;

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The last summand in Equation (2) consists of t open normal subgroups of index m^2 that do not fill in any of the three classes described: let us focus on one of these subgroups.

Lemma

If the conjecture is false, then there exists a finite nonabelian simple group T and a finite group H with the following properties:

1. $|H| = |T|^2$.
2. *H is not nilpotent, nor a direct product of two nonabelian simple groups, nor a direct product of a nilpotent group and a nonabelian simple group.*
3. *H contains a unique minimal normal sugroup N.*
4. *Either H/N is nilpotent, or there exists a finite nilpotent group X and a nonabelian simple group S such that $H/N \cong X \times S$. In the latter case $|T| \leq |S|$ and $\pi(S) = \pi(T)$.*

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If G is a perfect profinite group or a profinite group whose nonabelian composition factors are all alternating groups, then we can refine the previous result and provide a contradiction.

These evidences gives sufficient motivation to focus on finitely generated pro-p groups, as classifying normally ζ-reversible pro-p groups is the key to determine a classification of pronilpotent groups with this property.

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Let G be a pro-p group. Then G is normally ζ-reversible if and only if $G \cong \mathbb{Z}_p^n$.

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Conjecture

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Theorem (C. and González-Sánchez)

The last conjecture holds in the class of uniform pro-p group, for any odd prime p.

THANK YOU
FOR YOUR ATTENTION