Quasi-Sandwich Groups

(joint work with Gunnar Traustason - University of Bath)

Antonella Zaccardo

University of Naples *Federico II*

AGTA Workshop - Reinhold Baer Prize 2026

Left-Engel elements

Let G be a group. An element $a \in G$ is a *left Engel element* in G if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$[[[x, \underbrace{a], a], \cdots, a]}_{n(x)} = 1.$$

If $n(x)$ is bounded above by n, then we say that a is a *left n-Engel element* in G.
The symbol $L(G)$ will denote the set of all element verifying the general Engel condition, while $L_n(G)$ will is the set of all left n-Engel elements.

Remark

$$L_0(G) = \{1\}$$

Remark

$$L_0(G) = \{1\}$$

$$L_1(G) = Z(G)$$

Remark

$$L_0(G) = \{1\}$$

$$L_1(G) = Z(G)$$

$$L_2(G) = \{a \in G \mid \langle a \rangle^G \in \mathcal{A}\}$$

Remark

$$L_0(G) = \{1\}$$

$$L_1(G) = Z(G)$$

$$L_2(G) = \{a \in G \mid \langle a \rangle^G \in \mathcal{A}\} \subseteq HP(G)$$

Remark

$$L_0(G) = \{1\}$$

$$L_1(G) = Z(G)$$

$$L_2(G) = \{a \in G \mid \langle a \rangle^G \in \mathcal{A}\} \subseteq HP(G)$$

$$L_3(G) = ?$$

Remark

Let G be a group

$$a \in HP(G) \implies a \in L(G)$$

Remark

Let G be a group

$$a \in HP(G) \implies a \in L(G)$$
$$\impliedby ?$$

The central problem

The fundamental question is:

Does every Left 3-Engel element belong to the Hirsch–Plotkin radical?

or equivalently

If $a \in G$ is a left 3-Engel element, is $\langle a \rangle^G$ locally nilpotent?

Reformulating the problem

Let's recall a property that characterize left 3-Engel elements:

Lemma:

Let G be a group and $x \in G$, x is a left 3-Engel element if and only if $\langle x, x^y \rangle$ is nilpotent of class at most 2, for every $y \in G$.

Alireza Abdollahi, "Left 3-Engel elements in groups." Journal of Pure and Applied Algebra 188.1-3 (2004): 1-6.

Sandwich groups

In 2014, G. Traustason introduced a related class of groups:

Definition:

Let G be a group. We say that $X \subseteq G$ is a *sandwich set* in G if $\langle x, y^g \rangle$ is nilpotent of class at most 2 for all $x, y \in X$ and $g \in G$. A *sandwich group* is a group G that can be generated by a sandwich set X.



Gunnar Traustason, "Left 3-engel elements in groups of exponent 5", Journal of Algebra, 414:41–71, 2014.

Reformulating the problem

The original problem:

Reformulated version:

Reformulating the problem

The original problem:

Is every left 3-Engel element contained in $HP(G)$?

Reformulated version:

Reformulating the problem

The original problem:

Is every left 3-Engel element contained in $HP(G)$?

Reformulated version:

If a is a left 3-Engel element $\Rightarrow \langle a \rangle^G$ is a **sandwich group**

Reformulating the problem

The original problem:

Is every left 3-Engel element contained in $HP(G)$?

Reformulated version:

If a is a left 3-Engel element $\Rightarrow \langle a \rangle^G$ is a **sandwich group**

Let $x, y, g \in G$, we have that $\langle a^x, (a^y)^g \rangle^{x^{-1}} = \langle a, a^{ygx^{-1}} \rangle$ is of class at most 2, since a is a left 3-Engel. Hence, also $\langle a^x, (a^y)^g \rangle$ is nilpotent of class at most 2.

Reformulating the problem

The original problem:

Is every left 3-Engel element contained in $HP(G)$?

Reformulated version:

Is every finitely generated sandwich group nilpotent?

If a is a left 3-Engel element $\Rightarrow \langle a \rangle^G$ is a **sandwich group**

Let $x, y, g \in G$, we have that $\langle a^x, (a^y)^g \rangle^{x^{-1}} = \langle a, a^{ygx^{-1}} \rangle$ is of class at most 2, since a is a left 3-Engel. Hence, also $\langle a^x, (a^y)^g \rangle$ is nilpotent of class at most 2.

Sandwich groups of rank 3



Gunnar Traustason, "Left 3-engel elements in groups of exponent 5", Journal of Algebra, 414:41–71, 2014.

Theorem:

Any 3-generator sandwich group is nilpotent.

Sandwich groups of exponent 5



Gunnar Traustason, "Left 3-engel elements in groups of exponent 5", Journal of Algebra, 414:41–71, 2014.

Theorem:

Let G be a group of exponent 5 and let $a \in G$ be a left 3-Engel. We then have that a is in the Hirsch-Plotkin radical of G.

Sandwich groups of exponent 60



Gareth Tracey and Gunnar Traustason, "Left 3-Engel elements in groups of exponent 60", Int. J. Algebra and Comp. , 28. No. 4 (2018), 673–695.

Theorem:

Let G be any group and let x be a left 3-Engel in G of order dividing 60. Suppose furthermore that $\langle x \rangle^G$ has no elements of order 8, 9 or 25. Then x is contained in the Hirsh-Plotkin radical of G.

Left 3-Engel elements of finite order: the odd case



Enrico Jabara and Gunnar Traustason, "Left 3-Engel elements of odd order in groups", Proc. Am. Math. Soc. , 147 (2019), 1921-1927.

Theorem:

Let $G = \langle x_1, \ldots, x_r \rangle$ be a sandwich group where $x_1, \ldots, x_r$ are of odd order. Then G is nilpotent.

Left 3-Engel elements of finite order: the odd case



Enrico Jabara and Gunnar Traustason, "Left 3-Engel elements of odd order in groups", Proc. Am. Math. Soc. , 147 (2019), 1921-1927.

Theorem:

Let $G = \langle x_1, \ldots, x_r \rangle$ be a sandwich group where $x_1, \ldots, x_r$ are of odd order. Then G is nilpotent.

Corollary:

Let G be any group and let $a \in G$ be a left 3-Engel element of odd order. Then a belongs to the Hirsch-Plotkin radical of G.

Left 3-Engel elements of finite order

Lemma:

Let π be a set of primes. Suppose that for all groups G and all primes $p \in \pi$ we have that all left 3-Engel elements of order p in G are contained in $HP(G)$. It then follows that for all groups G, all left 3-Engel elements in G that are π-elements are in $HP(G)$.

Thus dealing with left 3-Engel elements of finite order reduces to working with sandwich groups generated by elements of prime order

The case of prime 2

What about left 3-Engel elements of order 2?

The case of prime 2

What about left 3-Engel elements of order 2?

Theorem:

Let x be a left 3-Engel involution in a group G. If $\langle x \rangle^G$ has no elements of order 8, then $\langle x \rangle^G$ is locally nilpotent.

What about left 3-Engel elements of order 2?

Theorem:

Let x be a left 3-Engel involution in a group G. If $\langle x \rangle^G$ has no elements of order 8, then $\langle x \rangle^G$ is locally nilpotent.


Anastasia Hadjievangelou and Gunnar Traustason, “Left 3-Engel elements in locally finite 2-groups”, Comm. Algebra, 49 No. 11 (2021), 4869–4882.


Anastasia Hadjievangelou, Marialaura Noce and Gunnar Traustason, "Locally finite p-groups with a left 3-Engel element whose normal closure is not nilpotent", Int. J. Alg. Comp. , 31 No. 1, (2021), 135-160.

Quasi-sandwich groups

Definition [G. Traustason, AZ (2026)]:

Let G be a group. We say that $X \subseteq G$ is a *quasi-sandwich set* in G if $\langle x, y^g \rangle$ is nilpotent of class at most 2 for all $x, y \in X$ **such that** $\boldsymbol{x \neq y}$ and $g \in G$.
Moreover, a *quasi-sandwich group* is a group G that can be generated by a quasi-sandwich set X.

The problem we address



Enrico Jabara and Gunnar Traustason, "Left 3-Engel elements of odd order in groups", Proc. Am. Math. Soc. , 147 (2019), 1921-1927.

Proposition:

Let $G = \langle x_1, \ldots, x_r \rangle$ be a sandwich group where $x_1, \ldots, x_r$ are of odd order. Then G is nilpotent.

The aim of our work is to investigate whether the theorem stated above can be generalized to **quasi-sandwich groups.**

Groups of rank 3 generated by quasi-sandwiches of odd order

Theorem [G. Traustason, AZ (2026)]:

Let $G = \langle x, y, z \rangle$ a quasi-sandwich group and assume that $o(x), o(y)$ and $o(z)$ are odd. Then G is nilpotent of class at most 3.

Groups of rank 3 generated by quasi-sandwich involutions

Theorem [G. Traustason, AZ (2026)]:

Let $F = \langle x, y, z \rangle$ be the free quasi-sandwich group of rank 3 generated by quasi-sandwiches of order 2. Let $a = z^{xy}z$, $b = z^{yx}z$ and $c = z^{xyxy}abz^{-1}$, we have

$$\gamma_{3+2k}(F) = \langle a^{2^k} b^{2^k}, [a^{2^k}, y] \rangle \gamma_{4+2k}(F)$$

$$\gamma_{4+2k}(F) = \langle a^{2^{k+1}}, [a^{2^k} b^{2^k}, y, c^{2^k}] \rangle \gamma_{3+2(k+1)}(F).$$

Moreover the ranks for the lower central factors are $r(F/\gamma_2(F)) = r(\gamma_2(F)/\gamma_3(F)) = 3$, $r(\gamma_{3+2k}(F)/\gamma_{4+2k}(F)) = 2$ and $r(\gamma_{4+2k}(F)/\gamma_{3+2(k+1)}(F)) = 3$, for $k \geq 0$.

Finitely generated quasi-sandwich groups

Lemma [G. Traustason, AZ (2026)]:

Let G be a group and $X \subseteq G$ a quasi-sandwich set in G where $o(x)$ is an odd number for all $x \in X$. Let $x, y \in X$, then $X \cup \{[x, y]\}$ is also a quasi-sandwich set in G and $o([x, y])$ is an odd number.

Finitely generated quasi-sandwich groups

Theorem [G. Traustason, AZ (2026)]:

Let $G = \langle x_1, \ldots, x_r \rangle$ be a quasi-sandwich group. If $o(x_i)$ is odd for every $i \in \{1, \ldots, r\}$, then G is nilpotent.

Thanks for your attention!