SET-THEORETIC SOLUTIONS OF THE YANG–BAXTER EQUATION FROM INVERSE BRACES

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In this talk:

- We review algebraic structures which give rise to set-theoretic solutions of the Yang–Baxter equation.
- Building on these structures, we extend this framework to the broader setting of inverse braces, highlighting their distinguishing properties and identifying sufficient conditions for generating solutions.
- We complement the main results with examples and conclude with open problems and possible directions for future research.

Based on some results contained in



F. Catino, M. Mazzotta, S.T., *Set-theoretic solutions of the Yang–Baxter equation from Inverse braces*, Semigroup Forum (2026)
https://doi.org/10.1007/s00233-026-10679-7

Definition

A set-theoretic solution of the YBE, briefly solution, is a pair (X, r) where X is a set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

In 1992, Drinfel'd[1] posed the problem of classifying all solutions, as a simplification of the original problem introduced by Yang[2] and Baxter[3] in Statistical Mechanics.

Identifying all the solutions to this equation still represents a key challenge, despite several studies on the matter.

[1] V. G. Drinfel'd, *On some unsolved problems in quantum group theory*, 1992
[2] C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, 1967
[3] R. J. Baxter, *Partition function of the eight-vertex lattice model*, 1972

Definition

A brace is a triple $(B, +, \circ)$ where $(B, +)$ is an abelian group, $(B, \circ)$ is a group and

$$\forall\, a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$$

Example. If $(R, +, \cdot)$ is a radical ring, i.e., the adjoint monoid $(R, \circ)$ of R is a group, where $a \circ b = a + ab + b$ for all $a, b \in R$, then $(R, +, \circ)$ is a brace.

Braces[4] **(2007)**

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Skew braces[5] **(2017)**

$(B, +)$ is an arbitrary group

→

Weak braces[6] **(2022)**

Groups are replaced by inverse semigroups

[4] W. Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, 2007

[5] L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, 2017

[6] F. Catino, M. Mazzotta, M.M. Miccoli, P. Stefanelli, *Set-theoretic solutions of the Yang-Baxter equation associated to weak braces*, 2022

Definition

A semigroup S is said to be inverse if, for all $a \in S$, there exists a unique $a^{-1} \in S$ such that

$$a = aa^{-1}a \quad \text{and} \quad a^{-1} = a^{-1}aa^{-1}.$$

Every group is an inverse semigroup.

In any inverse semigroup S, the following statements hold:

- $\forall\, a, b \in S \qquad (ab)^{-1} = b^{-1}a^{-1} \quad$ and $\quad (a^{-1})^{-1} = a$;
- $\mathsf{E}(S) = \{aa^{-1}, a^{-1}a \mid a \in S\}$.

[7] M. V. Lawson, *Inverse semigroups*, 1998

Definition

A triple $(S, +, \circ)$ is a weak (left) brace if $(S, +)$ and $(S, \circ)$ are inverse semigroups and the following relations hold

$$\begin{aligned} &\forall\, a, b, c \in S && a \circ (b + c) = a \circ b - a + a \circ c, \\ &\forall\, a \in S && a \circ a^{-} = -a + a. \end{aligned}$$

Every weak brace $(S, +, \circ)$ gives rise to a solution, defined, for all $a, b \in S$, by

$$r(a, b) = \big(-a + a \circ b,\ (-a + a \circ b)^{-} \circ a \circ b \big).$$

Example. If $(S, +)$ is a Clifford semigroup, then $(S, +, +)$ is a weak brace.

QUESTION

What if we drop the identity

$$a \circ a^{-} = -a + a?$$

Definition

A triple $(S, +, \circ)$ is an inverse brace if $(S, +)$ and $(S, \circ)$ are inverse semigroups and

$$\forall\, a, b, c \in S \qquad a \circ (b + c) = a \circ b - a + a \circ c.$$

Every weak brace is clearly an inverse brace.

- Consider the group $(\mathbb{Z}_2, +)$ and the operation $\circ$ defined by $0 \circ 0 = 0$ and $1 \circ 1 = 1 \circ 0 = 0 \circ 1 = 1$. Then $(\mathbb{Z}_2, +, \circ)$ is an inverse brace.
- Let $(R, +, \cdot)$ be a ring and define the adjoint operation $\circ$ on R by setting

$$a \circ b := a + ab + b \qquad \forall\, a, b \in R.$$

If $(R, \circ)$ is an inverse semigroup, then $(R, +, \circ)$ is an inverse brace.

A CHARACTERIZATION OF INVERSE BRACES

The following result provides a characterization of inverse braces that also appears in Wang[8].

Theorem

Let $(S, +)$ and $(S, \circ)$ be two inverse semigroups. Then $(S, +, \circ)$ is an inverse brace if and only if the following hold:

- for all $a \in S$, the map $\lambda_a : S \to S, \quad b \mapsto -a + a \circ b$ is a semigroup endomorphism of $(S, +)$;
- the map $\lambda : (S, \circ) \to \mathsf{End}(S, +), \quad a \mapsto \lambda_a$ is a semigroup homomorphism.

[8] S. Wang, *Some Characterizations of Weak Left Braces*, 2026

In any weak brace $(S, +, \circ)$, one has that $\mathsf{E}(S, +) = \mathsf{E}(S, \circ)$. Moreover, $(S, +)$ is necessarily Clifford.

Proposition.

Let $(S, +, \circ)$ be an inverse brace. Then

- $\mathsf{E}(S, +) \subseteq \mathsf{E}(S, \circ)$;
- $\mathsf{E}(S, +) = \mathsf{E}(S, \circ)$ if and only if $(S, +, \circ)$ is a weak brace.

For every inverse brace $(S, +, \circ)$ and $a \in S$, we denote by σ_a the map

$$\sigma_a : S \to S, \quad b \mapsto a \circ \left(a^- + b\right).$$

In any weak brace S, one has that

$$\sigma_a(b) = a \circ a^- - a + a \circ b = -a + a - a + a \circ b = \lambda_a(b) \qquad \forall\, a, b \in S.$$

THIS IDENTITY DOES NOT HOLD IN THE CASE OF INVERSE BRACES.

Proposition

If $(S, +, \circ)$ is an inverse brace, then, for all $a, b \in S$,

$$\sigma_a(b) = a \circ a^- + \lambda_a(b) = a \circ a^- \circ \lambda_a(b).$$

Moreover, $\sigma_a = \lambda_a$, for every $a \in S$, if and only if S is a weak brace.

IS σ_a AN ENDOMORPHISM OF THE ADDITIVE STRUCTURE?

Example. Let $S = \{0, x, y\}$. Consider the commutative inverse monoids $(S, +)$ and $(S, \circ)$, both with identity 0, whose binary operations are given by

$+$	0	x	y
0	0	x	y
x	x	x	y
y	y	y	x

and

$\circ$	0	x	y
0	0	x	y
x	x	x	y
y	y	y	y

Then $(S, +, \circ)$ is an inverse brace and $\sigma_y \notin \mathrm{End}(S, +)$.
Indeed, $\sigma_y(x + y) = \sigma_y(y) = y$, while $\sigma_y(x) + \sigma_y(y) = y + y = x$.

Proposition

Let $(S, +, \circ)$ be an inverse brace such that $(S, +)$ is a Clifford semigroup.
Then, for all $a \in S$, $\sigma_a \in \mathrm{End}(S, +)$ if and only if S is a weak brace.

IS σ STILL A HOMOMORPHISM?

Example. Consider the inverse brace $(\mathbb{Z}_3, +, \circ)$ induced by the adjoint operation on the ring $\mathbb{Z}_3$. In this example, the map σ is not a homomorphism since, for instance,

$$\sigma_{1\circ 2}(2) = 2, \text{ while } \sigma_1\sigma_2(2) = 1.$$

Proposition

Let $(S, +, \circ)$ be an inverse brace. Then, the map $\sigma : S \to S^S, a \mapsto \sigma_a$ is a semigroup homomorphism from $(S, \circ)$ to S^S if and only if

$$\forall a \in S, e \in \mathsf{E}(S, \circ) \quad \sigma_a(e) = a \circ e \circ a^- \tag{I}$$

Theorem [Catino, Mazzotta, S.T.]

Let $(S, +, \circ)$ be an inverse brace such that the map $\sigma : S \to S^S, a \mapsto \sigma_a$ is a homomorphism from the inverse semigroup $(S, \circ)$ to the monoid S^S, namely (I) is satisfied. Then the map $r_S : S \times S \to S \times S$ defined by

$$r_S(a, b) = \big(\sigma_a(b), \sigma_a(b)^- \circ a \circ b\big),$$

for all $a, b \in S$, is a solution if and only if the condition

$$\sigma_a(b) \circ \sigma_a(b)^- = \sigma_a\left(b \circ b^-\right), \tag{II}$$

is satisfied, for all $a, b \in S$.

EXAMPLES OF INVERSE BRACES YIELDING SOLUTIONS

- Inverse braces whose multiplicative structure is a semilattice always give rise to solutions, since the conditions (I) and (II) are trivially satisfied.
- Let $S = \{a, b, c, d\}$ and consider the following operations on S:

$+$	a	b	c	d
a	a	a	a	d
b	a	b	c	d
c	a	c	b	d
d	d	d	d	a

and

$\circ$	a	b	c	d
a	a	a	a	d
b	a	b	c	d
c	a	c	c	d
d	d	d	d	a

Then $(S, +, \circ)$ is an inverse brace satisfying (I) and (II). Thus the map r_S is a solution.

SOME OPEN PROBLEMS AND POSSIBLE RESEARCH DIRECTIONS

The following problems naturally arise from the study of inverse braces:

- The existence of solutions arising from inverse braces for which the map σ is not a homomorphism.
- Determining whether the additive structure of an inverse brace is a Clifford semigroup.
- Characterize the inverse braces $(S, +, \circ)$ and the parameters $z \in S$ for which it is possible to define a deformed[9] solution of the form $r_z(a, b) = (\sigma_a^z(b),\, \sigma_a^z(b)^- \circ a \circ b)$, with $\sigma_a^z(b) = a \circ a^- - a \circ z + a \circ b \circ z$.

[9] Mazzotta, M., Rybolowicz, B., Stefanelli, P., *Deformed solutions of the Yang- Baxter equation associated to dual weak braces*, 2025

THANK YOU FOR YOUR ATTENTION!