



Universal Localizations, Atiyah Conjectures and Graphs of Groups

AGTA Workshop 2026

Pablo Sánchez Peralta

Emmanuel College – DPMMS, University of Cambridge

October 27, 2026

Motivation

Goal: Understand our favorite group G.

Motivation

Goal: Understand our favorite group G.

$$\text{Classical group theory} \longleftrightarrow \text{Group algebras} \longleftrightarrow \text{Actions on a space}$$

Motivation

Goal: Understand our favorite group G.

$$\text{Classical group theory} \longleftrightarrow \text{Group algebras} \longleftrightarrow \text{Actions on a space}$$

In 1976, Atiyah posed the following seminal problem:

> *"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."*

Motivation

Goal: Understand our favorite group G.

$$\text{Classical group theory} \longleftrightarrow \text{Group algebras} \longleftrightarrow \text{Actions on a space}$$

In 1976, Atiyah posed the following seminal problem:

> *"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."*

For instance, if G is a residually finite group with finite classifying space X and we consider the action of G on the universal cover $\widetilde{X}$, by Lück's Approximation Theorem:

$$b_k^{(2)}(G) := \beta_k^{(2)}(\widetilde{X}, G) = \lim_{G_i \lhd_f G} \frac{b_k(G_i)}{|G : G_i|}$$

Motivation

Goal: Understand our favorite group G.

$$\text{Classical group theory} \longleftrightarrow \text{Group algebras} \longleftrightarrow \text{Actions on a space}$$

In 1976, Atiyah posed the following seminal problem:

> *"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."*

For instance, if G is a residually finite group with finite classifying space X and we consider the action of G on the universal cover $\widetilde{X}$, by Lück's Approximation Theorem:

$$b_k^{(2)}(G) := \beta_k^{(2)}(\widetilde{X}, G) = \lim_{G_i \lhd_f G} \frac{b_k(G_i)}{|G : G_i|} \rightsquigarrow \text{Why should it be a rational number?}$$

L^2-Betti numbers and their connections

L^2-Betti numbers and their connections

Groups

- [Kielak '20] The group $A(\Gamma)$ with Γ a pentagon graph admits a map onto $\mathbb{Z}$ with finitely generated kernel but not finitely presented.
- [Jaikin-Zapirain '24] If $F_2 < F$ and $H < F$, then $\mathrm{rk}(F_2 \cap H) \leq \mathrm{rk}(H)$.
- [Kielak–Linton '24] One-relator groups with torsion are virtually free-by-cyclic.
- [Jaikin-Zapirain–Linton '25] One-relator groups are coherent.
- [Fisher–SP '26] Coherent virtually RFRS groups of cd 2 are virtually free-by-cyclic.

L^2-Betti numbers and their connections

Group Rings

- [Linnell '93, Friedl–Lück '19, Kielak–Linton '24] If M^3 is a connected 3-manifold with $\pi_1(M^3)$ torsion-free, then $\mathbb{C}[\pi_1(M^3)]$ is a domain.
- [Jaikin-Zapirain–Linton '25] Complex group algebras of one-relator groups are coherent.

L^2-Betti numbers and their connections

Group Rings

- [Linnell '93, Friedl–Lück '19, Kielak–Linton '24] If M^3 is a connected 3-manifold with $\pi_1(M^3)$ torsion-free, then $\mathbb{C}[\pi_1(M^3)]$ is a domain.
- [Jaikin-Zapirain–Linton '25] Complex group algebras of one-relator groups are coherent.

Topology

- [Sauer–Thom '10] The Hopf conjecture holds for any closed aspherical manifold with poly-surface fundamental group.
- [Agol '13] Every closed hyperbolic 3-manifold virtually fibers over the circle ([Kielak '20]).
- [SP '26] If a knot group G_K maps onto $G_{K'}$, then $\text{genus}(K) \geq \text{genus}(K')$.

Atiyah's problem

Atiyah

"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."

Atiyah's problem

Atiyah

"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."

- [Austin '13] The set of real values $\{\beta_k^{(2)}(X, G)\}$ is uncountable.

Atiyah's problem

Atiyah

"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."

- [Austin '13] The set of real values $\{\beta_k^{(2)}(X, G)\}$ is uncountable.
- [Grabowski '14] Any non-negative real number is realized.

Atiyah's problem

Atiyah

"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."

- [Austin '13] The set of real values $\{\beta_k^{(2)}(X, G)\}$ is uncountable.
- [Grabowski '14] Any non-negative real number is realized.

Problem: We allow groups with very large finite subgroups like the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.

Atiyah's problem

Atiyah

"Give examples for which $\beta_k^{(2)}(X, G)$ are not integral or even perhaps irrational numbers."

- [Austin '13] The set of real values $\{\beta_k^{(2)}(X, G)\}$ is uncountable.
- [Grabowski '14] Any non-negative real number is realized.

Problem: We allow groups with very large finite subgroups like the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.

$\rightsquigarrow$ Restrict the size of finite subgroups: Strong Atiyah Conjecture

Statement

Definition

Given a group G, the *least common multiple of* G is the (possibly infinite) value

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|H| : H \leq G \text{ and } H \text{ is finite}\}.$$

Statement

Definition

Given a group G, the *least common multiple of* G is the (possibly infinite) value

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|H| : H \leq G \text{ and } H \text{ is finite}\}.$$

Examples:

Statement

Definition

Given a group G, the *least common multiple of* G is the (possibly infinite) value

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|H| : H \leq G \text{ and } H \text{ is finite}\}.$$

Examples:

1. If G is a finite group, $\operatorname{lcm}(G) = |G|$.

Statement

Definition

Given a group G, the *least common multiple of* G is the (possibly infinite) value

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|H| : H \leq G \text{ and } H \text{ is finite}\}.$$

Examples:

1. If G is a finite group, $\operatorname{lcm}(G) = |G|$.
2. If G is a torsion-free group, $\operatorname{lcm}(G) = 1$.

Statement

Definition

Given a group G, the *least common multiple of* G is the (possibly infinite) value

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|H| : H \leq G \text{ and } H \text{ is finite}\}.$$

Examples:

1. If G is a finite group, $\operatorname{lcm}(G) = |G|$.
2. If G is a torsion-free group, $\operatorname{lcm}(G) = 1$.
3. If $G = C_2 * C_3$, $\operatorname{lcm}(G) = 6$.

Statement

Definition

Given a group G, the *least common multiple of* G is the (possibly infinite) value

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|H| : H \leq G \text{ and } H \text{ is finite}\}.$$

Examples:

1. If G is a finite group, $\operatorname{lcm}(G) = |G|$.
2. If G is a torsion-free group, $\operatorname{lcm}(G) = 1$.
3. If $G = C_2 * C_3$, $\operatorname{lcm}(G) = 6$. More generally,

$$\operatorname{lcm}(A * B) = \operatorname{lcm}\{\operatorname{lcm}(A), \operatorname{lcm}(B)\}.$$

Statement & status

Conjecture (Strong Atiyah Conjecture)

Let G be a group with finite $\mathrm{lcm}(G)$*. Then for every finitely presented* $\mathbb{C}[G]$*-module* M

$$\dim_{\mathcal{U}(G)} \left(\mathcal{U}(G) \otimes_{\mathbb{C}[G]} M\right) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

Statement & status

Conjecture (Strong Atiyah Conjecture)

Let G be a group with finite $\mathrm{lcm}(G)$*. Then for every finitely presented* $\mathbb{C}[G]$*-module* M

$$\dim_{\mathcal{U}(G)} \left(\mathcal{U}(G) \otimes_{\mathbb{C}[G]} M\right) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

We say that $G \in \mathcal{SAC}$ if the group G satisfies the Strong Atiyah Conjecture.

Statement & status

Conjecture (Strong Atiyah Conjecture)

Let G be a group with finite $\mathrm{lcm}(G)$*. Then for every finitely presented* $\mathbb{C}[G]$*-module* M

$$\dim_{\mathcal{U}(G)}\left(\mathcal{U}(G)\otimes_{\mathbb{C}[G]} M\right) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

We say that $G \in \mathcal{SAC}$ if the group G satisfies the Strong Atiyah Conjecture.

The Strong Atiyah Conjecture is known to hold for large classes of groups.

Statement & status

Conjecture (Strong Atiyah Conjecture)

Let G be a group with finite $\mathrm{lcm}(G)$*. Then for every finitely presented* $\mathbb{C}[G]$*-module* M

$$\dim_{\mathcal{U}(G)}\left(\mathcal{U}(G)\otimes_{\mathbb{C}[G]} M\right) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

We say that $G \in \mathcal{SAC}$ if the group G satisfies the Strong Atiyah Conjecture.

The Strong Atiyah Conjecture is known to hold for large classes of groups.

But we know very little about its behaviour...

Open questions

Natural Questions:

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G$

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.
2. Let $H \in \mathcal{SAC}$ and $H <_{\mathsf{f}} G$

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.
2. Let $H \in \mathcal{SAC}$ and $H <_{\mathrm{f}} G \overset{?}{\Rightarrow} G \in \mathcal{SAC}$.

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.
2. Let $H \in \mathcal{SAC}$ and $H <_f G \overset{?}{\Rightarrow} G \in \mathcal{SAC}$.
3. Let $G_1, G_2 \in \mathcal{SAC}$,

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.
2. Let $H \in \mathcal{SAC}$ and $H <_f G \overset{?}{\Rightarrow} G \in \mathcal{SAC}$.
3. Let $G_1, G_2 \in \mathcal{SAC}$,
$$\overset{?}{\Rightarrow} G_1 - \text{by} - G_2 \in \mathcal{SAC}.$$

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.
2. Let $H \in \mathcal{SAC}$ and $H <_f G \overset{?}{\Rightarrow} G \in \mathcal{SAC}$.
3. Let $G_1, G_2 \in \mathcal{SAC}$,

$$\overset{?}{\Rightarrow} G_1 - \text{by} - G_2 \in \mathcal{SAC}.$$

$$\overset{?}{\Rightarrow} G_1 \times G_2 \in \mathcal{SAC}.$$

Open questions

Natural Questions:

1. Let $G \in \mathcal{SAC}$ and $H \leq G \overset{?}{\Rightarrow} H \in \mathcal{SAC}$.
2. Let $H \in \mathcal{SAC}$ and $H <_f G \overset{?}{\Rightarrow} G \in \mathcal{SAC}$.
3. Let $G_1, G_2 \in \mathcal{SAC}$,

$$\overset{?}{\Rightarrow} G_1 - \text{by} - G_2 \in \mathcal{SAC}.$$

$$\overset{?}{\Rightarrow} G_1 \times G_2 \in \mathcal{SAC}.$$

$$\overset{?}{\Rightarrow} G_1 * G_2 \in \mathcal{SAC}.$$

Results

Theorem [SP'25]

Let G be the fundamental group of a graph of groups $\mathcal{G}_\Gamma = (G_v, G_e)$ such that every vertex group $G_v \in \mathcal{SAC}$ and every edge group G_e is finite. If $\operatorname{lcm}(G)$ is finite, then $G \in \mathcal{SAC}$.

Results

Theorem [SP'25]

Let G be the fundamental group of a graph of groups $\mathcal{G}_\Gamma = (G_v, G_e)$ such that every vertex group $G_v \in \mathcal{SAC}$ and every edge group G_e is finite. If $\mathrm{lcm}(G)$ is finite, then $G \in \mathcal{SAC}$.

Corollary [SP'25]

For any two groups $A, B \in \mathcal{SAC}$ the free product $A * B \in \mathcal{SAC}$.

L^2-theory and L^2-Betti numbers

The group algebra

We will study G via its group algebra $\mathbb{C}[G]$:

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \text{ and only finitely many } a_g \neq 0 \right\}.$$

The group algebra

We will study G via its group algebra $\mathbb{C}[G]$:

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \text{ and only finitely many } a_g \neq 0 \right\}.$$

Goal: Define a meaningful notion of dimension for $\mathbb{C}[G]$-modules.

The group algebra

We will study G via its group algebra $\mathbb{C}[G]$:

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \text{ and only finitely many } a_g \neq 0 \right\}.$$

Goal: Define a meaningful notion of dimension for $\mathbb{C}[G]$-modules.

If G is a finite group, we can set

$$\dim_G(M) := \frac{1}{|G|} \dim_{\mathbb{C}}(M).$$

However, if G is an infinite group, this formula collapses.

The group algebra

We will study G via its group algebra $\mathbb{C}[G]$:

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \text{ and only finitely many } a_g \neq 0 \right\}.$$

Goal: Define a meaningful notion of dimension for $\mathbb{C}[G]$-modules.

If G is a finite group, we can set

$$\dim_G(M) := \frac{1}{|G|} \dim_{\mathbb{C}}(M).$$

However, if G is an infinite group, this formula collapses.

Solution: consider $\mathbb{C}[G] \hookrightarrow \mathcal{B}(\ell^2(G))$.

The Hilbert space of a group

There is a natural inner product in $\mathbb{C}[G]$:

$$\left\langle \sum_{g \in G} a_g g, \sum_{g \in G} b_g g \right\rangle = \sum_{g \in G} a_g \overline{b_g}.$$

The Hilbert space of a group

There is a natural inner product in $\mathbb{C}[G]$:

$$\left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g \right\rangle = \sum_{g\in G} a_g \overline{b_g}.$$

Taking the completion yields the Hilbert space $\ell^2(G)$:

$$\ell^2(G) := \left\{ \sum_{g\in G} a_g g : a_g \in \mathbb{C} \text{ and } \sum_{g\in G} |a_g|^2 < \infty \right\}.$$

The Hilbert space of a group

There is a natural inner product in $\mathbb{C}[G]$:

$$\left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g \right\rangle = \sum_{g\in G} a_g \overline{b_g}.$$

Taking the completion yields the Hilbert space $\ell^2(G)$:

$$\ell^2(G) := \left\{ \sum_{g\in G} a_g g : a_g \in \mathbb{C} \text{ and } \sum_{g\in G} |a_g|^2 < \infty \right\}.$$

$G \curvearrowright \ell^2(G)$ via right and left multiplication:

$$\mathbb{C}[G] \overset{\text{right action}}{\hookrightarrow} \mathcal{B}(\ell^2(G)).$$

Operator algebras

The *group von Neumann algebra* of G is the ring

$$\mathcal{N}(G) := \left\{ T \in \mathcal{B}(\ell^2(G)) : T(g \cdot v) = g \cdot T(v), \forall g \in G, \forall v \in \ell^2(G) \right\}.$$

Operator algebras

The *group von Neumann algebra* of G is the ring

$$\mathcal{N}(G) := \left\{ T \in \mathcal{B}(\ell^2(G)) : T(g \cdot v) = g \cdot T(v), \forall g \in G, \forall v \in \ell^2(G) \right\}.$$

Note that $\mathbb{C}[G] \subseteq \mathcal{N}(G)$ since $g_1 \cdot (h \cdot g_2) = (g_1 \cdot h) \cdot g_2$.

Operator algebras

The *group von Neumann algebra* of G is the ring

$$\mathcal{N}(G) := \left\{ T \in \mathcal{B}(\ell^2(G)) : T(g \cdot v) = g \cdot T(v), \forall g \in G, \forall v \in \ell^2(G) \right\}.$$

Note that $\mathbb{C}[G] \subseteq \mathcal{N}(G)$ since $g_1 \cdot (h \cdot g_2) = (g_1 \cdot h) \cdot g_2$.

The *ring of unbounded operators affiliated* to G is the Ore localization

$$\mathcal{U}(G) := \mathrm{Ore}(\mathcal{N}(G)).$$

Operator algebras

The *group von Neumann algebra* of G is the ring

$$\mathcal{N}(G) := \left\{ T \in \mathcal{B}(\ell^2(G)) : T(g \cdot v) = g \cdot T(v), \forall g \in G, \forall v \in \ell^2(G) \right\}.$$

Note that $\mathbb{C}[G] \subseteq \mathcal{N}(G)$ since $g_1 \cdot (h \cdot g_2) = (g_1 \cdot h) \cdot g_2$.

The *ring of unbounded operators affiliated* to G is the Ore localization

$$\mathcal{U}(G) := \mathrm{Ore}(\mathcal{N}(G)).$$

Thus, every $u \in \mathcal{U}(G)$ can be written as

$$u = s^{-1}t \quad \text{where both } s, t \in \mathcal{N}(G)$$

and s is not a zero-divisor.

Operator algebras: examples

Examples:

1. If G is a finite group: We have $\mathbb{C}[G] = \mathbb{C}^{|G|}$.

Operator algebras: examples

Examples:

1. If G is a finite group: We have $\mathbb{C}[G] = \mathbb{C}^{|G|}$.

$$\mathbb{C}[G] = \ell^2(G)$$

$$\mathbb{C}[G] = \mathcal{N}(G) = \mathcal{U}(G)$$

Operator algebras: examples

Examples:

1. If G is a finite group: We have $\mathbb{C}[G] = \mathbb{C}^{|G|}$.

$$\mathbb{C}[G] = \ell^2(G)$$

$$\mathbb{C}[G] = \mathcal{N}(G) = \mathcal{U}(G)$$

2. If $G = \langle t \rangle \cong \mathbb{Z}$: We have $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and we can identify

$$\ell^2(\mathbb{Z}) \xleftrightarrow{\text{Fourier transform}} L^2([0,1])$$

Operator algebras: examples

Examples:

1. If G is a finite group: We have $\mathbb{C}[G] = \mathbb{C}^{|G|}$.

$$\mathbb{C}[G] = \ell^2(G)$$

$$\mathbb{C}[G] = \mathcal{N}(G) = \mathcal{U}(G)$$

2. If $G = \langle t \rangle \cong \mathbb{Z}$: We have $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and we can identify

$$\ell^2(\mathbb{Z}) \xleftrightarrow{\text{Fourier transform}} L^2([0,1])$$

Under this identification:

$$\mathcal{N}(G) = L^\infty([0,1]) \text{ and } \mathcal{U}(G) = \mathcal{M}([0,1]).$$

The L^2-dimension

The L^2-Solution: There is a well-defined dimension over arbitrary $\mathcal{U}(G)$-modules satisfying:

1. (Normalization) $\dim_{\mathcal{U}(G)}(\mathcal{U}(G)) = 1$
2. (Additivity) For any exact sequence $0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow 0$,

$$\dim_{\mathcal{U}(G)}(L_2) = \dim_{\mathcal{U}(G)}(L_1) + \dim_{\mathcal{U}(G)}(L_3)$$

The L^2-dimension

The L^2-Solution: There is a well-defined dimension over arbitrary $\mathcal{U}(G)$-modules satisfying:

1. (Normalization) $\dim_{\mathcal{U}(G)}(\mathcal{U}(G)) = 1$
2. (Additivity) For any exact sequence $0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow 0$,

$$\dim_{\mathcal{U}(G)}(L_2) = \dim_{\mathcal{U}(G)}(L_1) + \dim_{\mathcal{U}(G)}(L_3)$$

We get a dimension function on $\mathbb{C}[G]$ via the pullback:

$$\dim_G(M) := \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}[G]} M)$$

L^2-Betti numbers

Warning

Tensoring with $\mathcal{U}(G)$ loses information about injective maps.

L^2-Betti numbers

Warning

Tensoring with $\mathcal{U}(G)$ loses information about injective maps.

Definition

Let M be a $\mathbb{C}[G]$-module and k a non-negative integer. The *kth L^2-Betti number* of M is the (possibly infinite) value

$$\beta_k^G(M) := \dim_{\mathcal{U}(G)} \left(\mathrm{Tor}_k^{\mathbb{C}[G]}(\mathcal{U}(G), M) \right).$$

L^2-Betti numbers

Warning

Tensoring with $\mathcal{U}(G)$ loses information about injective maps.

Definition

Let M be a $\mathbb{C}[G]$-module and k a non-negative integer. The *kth L^2-Betti number* of M is the (possibly infinite) value

$$\beta_k^G(M) := \dim_{\mathcal{U}(G)} \left(\mathrm{Tor}_k^{\mathbb{C}[G]}(\mathcal{U}(G), M) \right).$$

The *kth L^2-Betti number* of G is defined as

$$b_k^{(2)}(G) := \beta_k^G(\mathbb{C}) = \dim_{\mathcal{U}(G)}(\mathrm{H}_k(G; \mathcal{U}(G))).$$

The Strong Atiyah Conjecture

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G]$

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M)$

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

   In our case, $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and $\mathcal{U}(G) = \mathcal{M}([0,1])$

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

   In our case, $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and $\mathcal{U}(G) = \mathcal{M}([0,1]) \rightsquigarrow \mathcal{D}(G) = \mathbb{C}(t)$.

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

   In our case, $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and $\mathcal{U}(G) = \mathcal{M}([0,1]) \rightsquigarrow \mathcal{D}(G) = \mathbb{C}(t)$.

$$\Rightarrow \beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}(t)} (\mathbb{C}(t) \otimes_{\mathbb{C}[G]} M))$$

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

   In our case, $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and $\mathcal{U}(G) = \mathcal{M}([0,1]) \rightsquigarrow \mathcal{D}(G) = \mathbb{C}(t)$.

$$\Rightarrow \beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}(t)} \underbrace{(\mathbb{C}(t) \otimes_{\mathbb{C}[G]} M)}_{\cong \mathbb{C}(t)^n})$$

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

   In our case, $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and $\mathcal{U}(G) = \mathcal{M}([0,1]) \rightsquigarrow \mathcal{D}(G) = \mathbb{C}(t)$.

$$\Rightarrow \beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}(t)} \underbrace{(\mathbb{C}(t) \otimes_{\mathbb{C}[G]} M)}_{\cong \mathbb{C}(t)^n}) = n$$

Examples

Conjecture (Strong Atiyah Conjecture)

For every finitely presented $\mathbb{C}[G]$-module M it holds that $\beta_0^G(M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Examples:

1. If G is a finite group, $\mathcal{U}(G) = \mathbb{C}[G] \Rightarrow \beta_0^G(M) = \frac{1}{|G|}\dim_{\mathbb{C}}(M) \in \frac{1}{|G|}\mathbb{Z}$.
2. If $G = \langle t \rangle \cong \mathbb{Z}$,

   $\rightarrow$ Use the Linnell ring: $\mathcal{D}(G) = \mathrm{Div}(\mathbb{C}[G], \mathcal{U}(G))$.

   In our case, $\mathbb{C}[G] = \mathbb{C}[t^{\pm 1}]$ and $\mathcal{U}(G) = \mathcal{M}([0,1]) \rightsquigarrow \mathcal{D}(G) = \mathbb{C}(t)$.

$$\Rightarrow \beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}(t)} \underbrace{(\mathbb{C}(t) \otimes_{\mathbb{C}[G]} M)}_{\cong \mathbb{C}(t)^n}) = n \in \mathbb{Z}.$$

Results

Theorem [SP'25]

Let G be the fundamental group of a graph of groups $\mathcal{G}_\Gamma = (G_v, G_e)$ such that every vertex group $G_v \in \mathcal{SAC}$ and every edge group G_e is finite. If $\operatorname{lcm}(G)$ is finite, then $G \in \mathcal{SAC}$.

Corollary [SP'25]

For any two groups $A, B \in \mathcal{SAC}$ the free product $A * B \in \mathcal{SAC}$.

Results

Theorem [SP'25]

Let G be the fundamental group of a graph of groups $\mathcal{G}_\Gamma = (G_v, G_e)$ such that every vertex group $G_v \in \mathcal{SAC}$ and every edge group G_e is finite. If $\mathrm{lcm}(G)$ is finite, then $G \in \mathcal{SAC}$.

Corollary [SP'25]

For any two groups $A, B \in \mathcal{SAC}$ the free product $A * B \in \mathcal{SAC}$.

Key Insight: For $G = A * B$ and torsion-free, we will use the graph of rings

$$R_G := \mathcal{D}(A) * \mathcal{D}(B).$$

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} (R_G \otimes_{\mathbb{C}[G]} M))$$

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} \underbrace{(R_G \otimes_{\mathbb{C}[G]} M)}_{=: \widetilde{M}})$$

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} \underbrace{(R_G \otimes_{\mathbb{C}[G]} M)}_{=: \widetilde{M}})$$

For $\widetilde{M}$ there is a short exact sequence

$$0 \longrightarrow R_G \otimes_{\mathbb{C}[A]} I_1 \oplus R_G \otimes_{\mathbb{C}[B]} I_2 \longrightarrow R_G^d \longrightarrow \widetilde{M} \longrightarrow 0.$$

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} \underbrace{(R_G \otimes_{\mathbb{C}[G]} M)}_{=: \widetilde{M}})$$

For $\widetilde{M}$ there is a short exact sequence

$$0 \longrightarrow R_G \otimes_{\mathbb{C}[A]} I_1 \oplus R_G \otimes_{\mathbb{C}[B]} I_2 \longrightarrow R_G^d \longrightarrow \widetilde{M} \longrightarrow 0.$$

If it holds that

$$\beta_0^G(M) = d - \left(\beta_0^A(I_1) + \beta_0^B(I_2)\right)$$

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} \underbrace{(R_G \otimes_{\mathbb{C}[G]} M)}_{=: \widetilde{M}})$$

For $\widetilde{M}$ there is a short exact sequence

$$0 \longrightarrow R_G \otimes_{\mathbb{C}[A]} I_1 \oplus R_G \otimes_{\mathbb{C}[B]} I_2 \longrightarrow R_G^d \longrightarrow \widetilde{M} \longrightarrow 0.$$

If it holds that

$$\beta_0^G(M) = d - \left(\underbrace{\beta_0^A(I_1)}_{\in \frac{1}{\mathrm{lcm}(A)}\mathbb{Z}} + \underbrace{\beta_0^B(I_2)}_{\in \frac{1}{\mathrm{lcm}(B)}\mathbb{Z}} \right)$$

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} \underbrace{(R_G \otimes_{\mathbb{C}[G]} M)}_{=: \widetilde{M}})$$

For $\widetilde{M}$ there is a short exact sequence

$$0 \longrightarrow R_G \otimes_{\mathbb{C}[A]} I_1 \oplus R_G \otimes_{\mathbb{C}[B]} I_2 \longrightarrow R_G^d \longrightarrow \widetilde{M} \longrightarrow 0.$$

If it holds that

$$\beta_0^G(M) = d - \Big(\underbrace{\beta_0^A(I_1)}_{\in \frac{1}{\mathrm{lcm}(A)}\mathbb{Z}} + \underbrace{\beta_0^B(I_2)}_{\in \frac{1}{\mathrm{lcm}(B)}\mathbb{Z}} \Big) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

□

Sketch proof

As before,

$$\beta_0^G(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{R_G} \underbrace{(R_G \otimes_{\mathbb{C}[G]} M)}_{=: \widetilde{M}})$$

For $\widetilde{M}$ there is a short exact sequence

$$0 \longrightarrow R_G \otimes_{\mathbb{C}[A]} I_1 \oplus R_G \otimes_{\mathbb{C}[B]} I_2 \longrightarrow R_G^d \longrightarrow \widetilde{M} \longrightarrow 0.$$

If it holds that

$$\beta_0^G(M) = d - \Big(\underbrace{\beta_0^A(I_1)}_{\in \frac{1}{\mathrm{lcm}(A)}\mathbb{Z}} + \underbrace{\beta_0^B(I_2)}_{\in \frac{1}{\mathrm{lcm}(B)}\mathbb{Z}} \Big) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

□

→ **Exactness is provided by the vanishing of** $\mathbf{Tor}_1^{R_G}(\mathcal{U}(G), \mathcal{U}(G))$.

Thank you for your attention!