Rational embeddings of continuous automatic groups

Francesco Matucci

AGTA workshop - Reinhold Baer Prize, Università di Napoli 2026

Collaborators



Jim Belk



Indira Chatterji



Collin Bleak



Davide Perego

Geometric action

Let G be a discrete group. An action of G on a metric space X is **geometric** if

1. each $g \in G$ is an isometry;
2. the quotient $G\backslash X$ is compact;
3. for each compact $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite.

Geometric action

Let G be a discrete group. An action of G on a metric space X is **geometric** if

1. each $g \in G$ is an isometry;
2. the quotient $G\backslash X$ is compact;
3. for each compact $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite.

Informally, this ensure that the "geometry" of the group is coarsely the same as the one of the space.

Geometric action

Let G be a discrete group. An action of G on a metric space X is **geometric** if

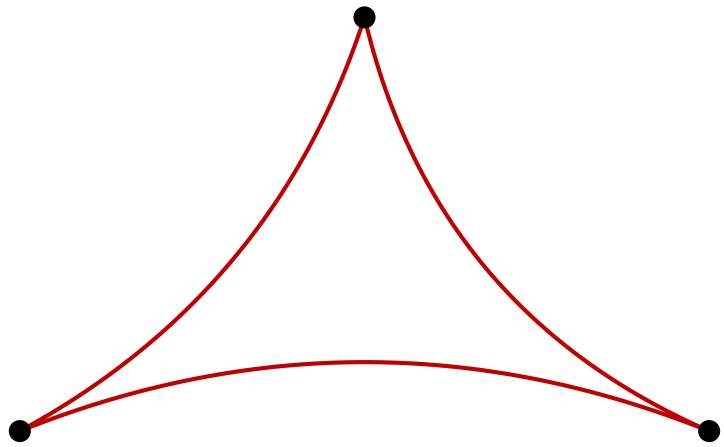
1. each $g \in G$ is an isometry;
2. the quotient $G\backslash X$ is compact;
3. for each compact $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite.

Informally, this ensure that the "geometry" of the group is coarsely the same as the one of the space.

Our main goal: represent this action using abstract machines in different ways.

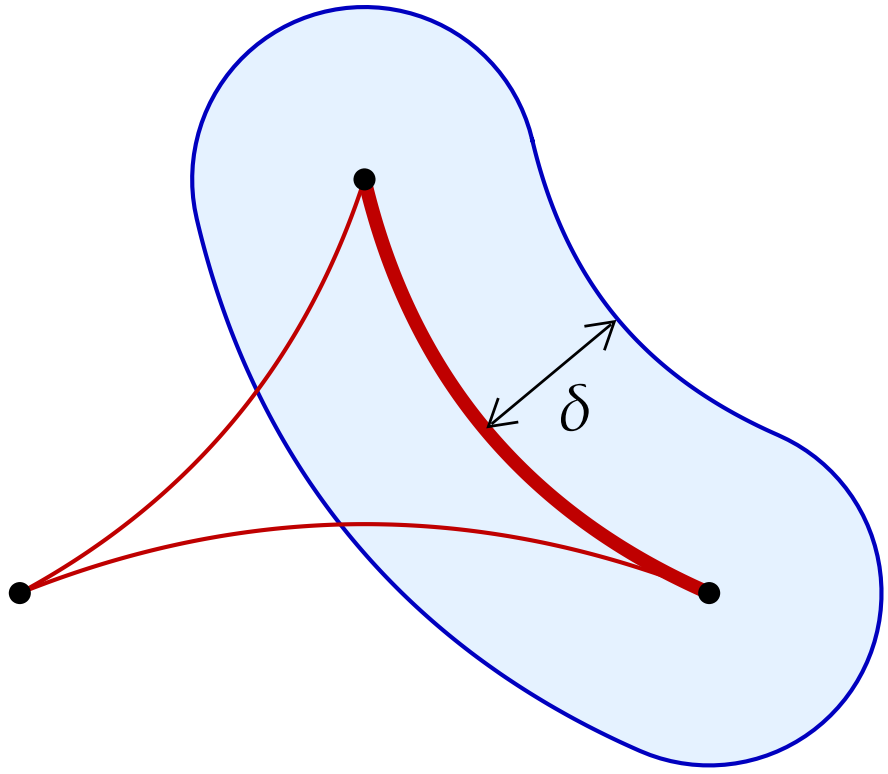
Hyperbolic groups

A locally finite graph X is **hyperbolic** if there exists $\delta \geq 0$ such that



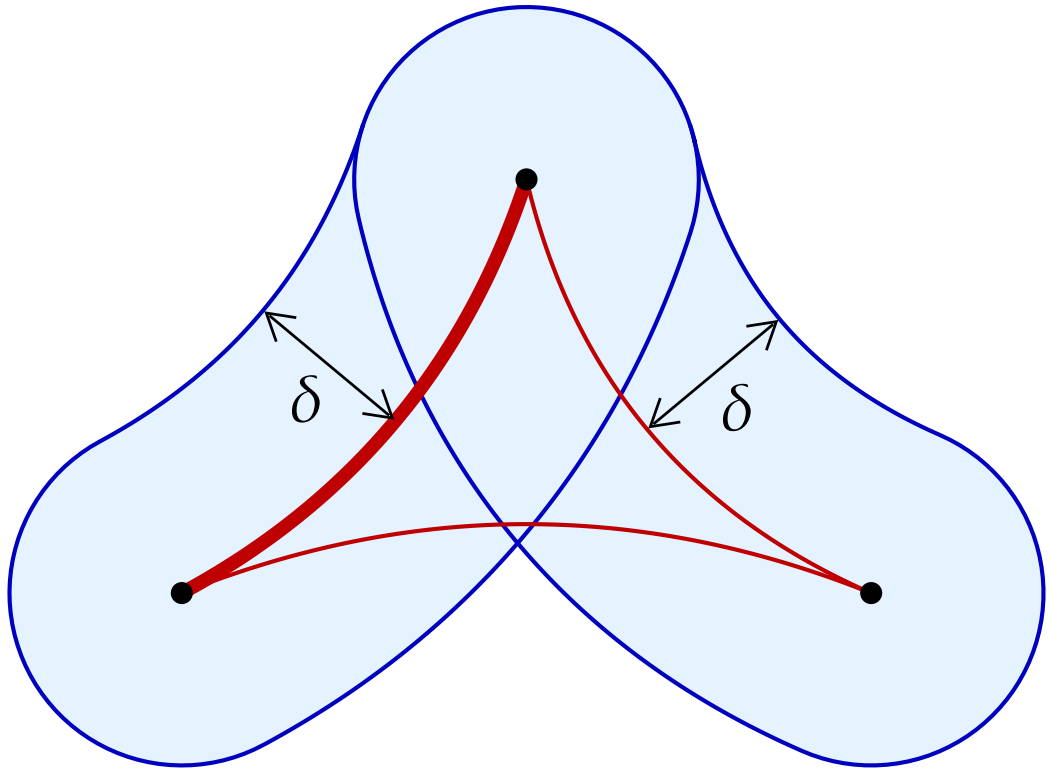
Hyperbolic groups

A locally finite graph X is **hyperbolic** if there exists $\delta \geq 0$ such that



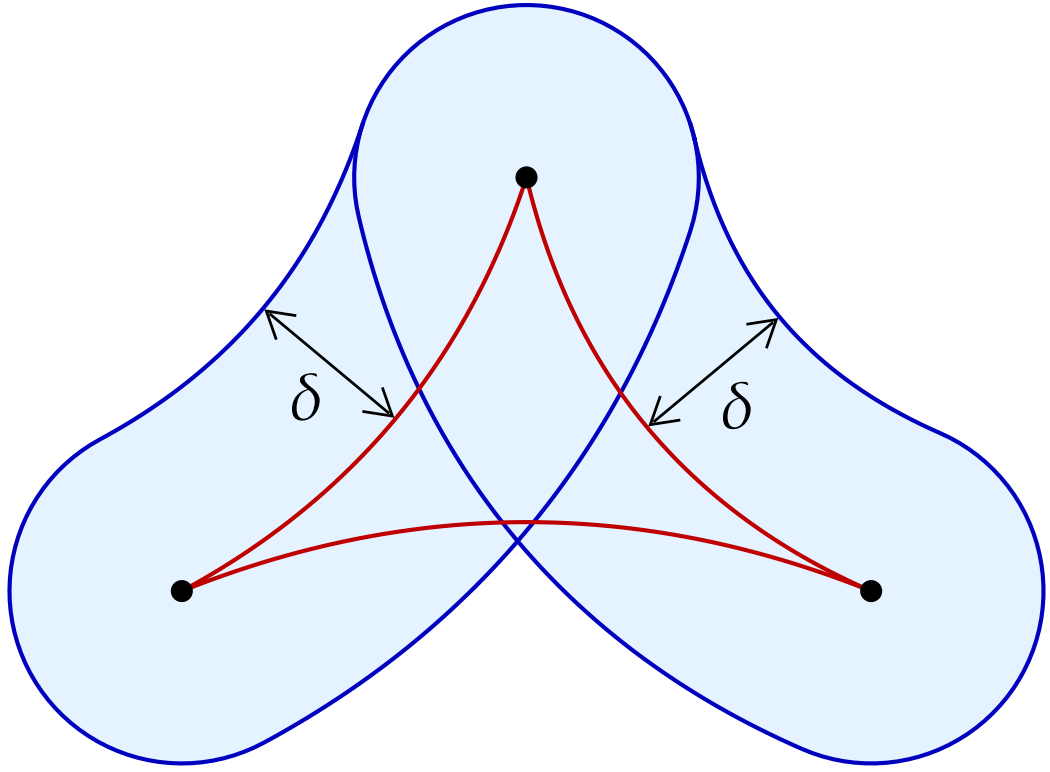
Hyperbolic groups

A locally finite graph X is **hyperbolic** if there exists $\delta \geq 0$ such that



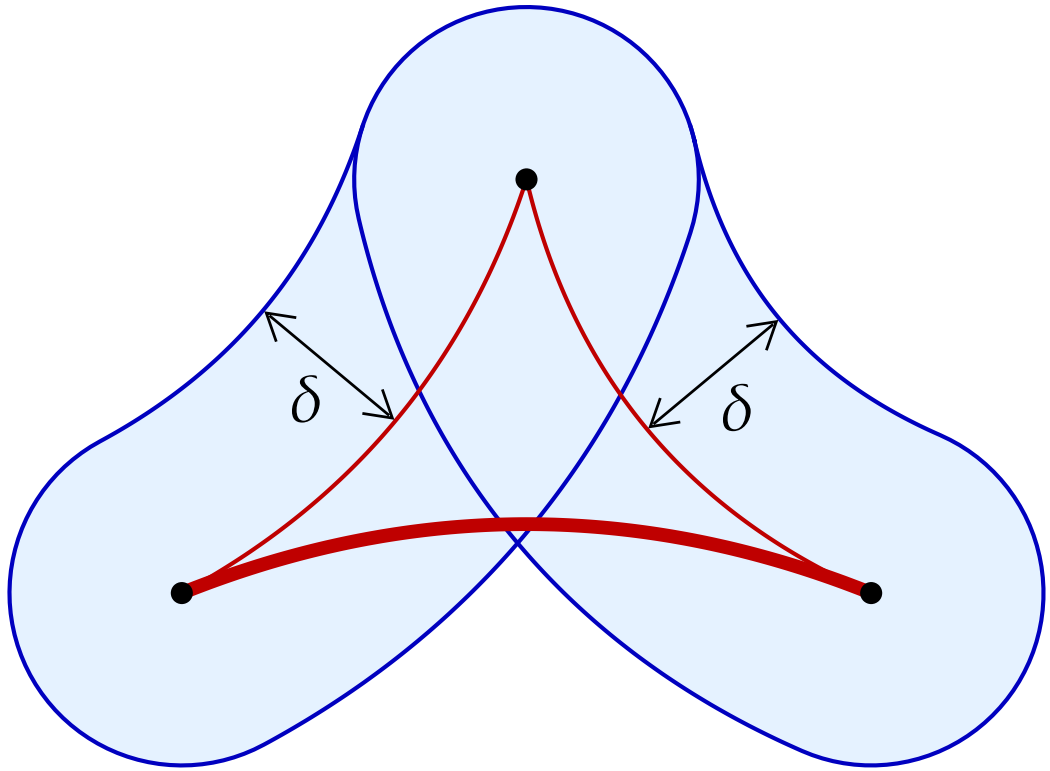
Hyperbolic groups

A locally finite graph X is **hyperbolic** if there exists $\delta \geq 0$ such that



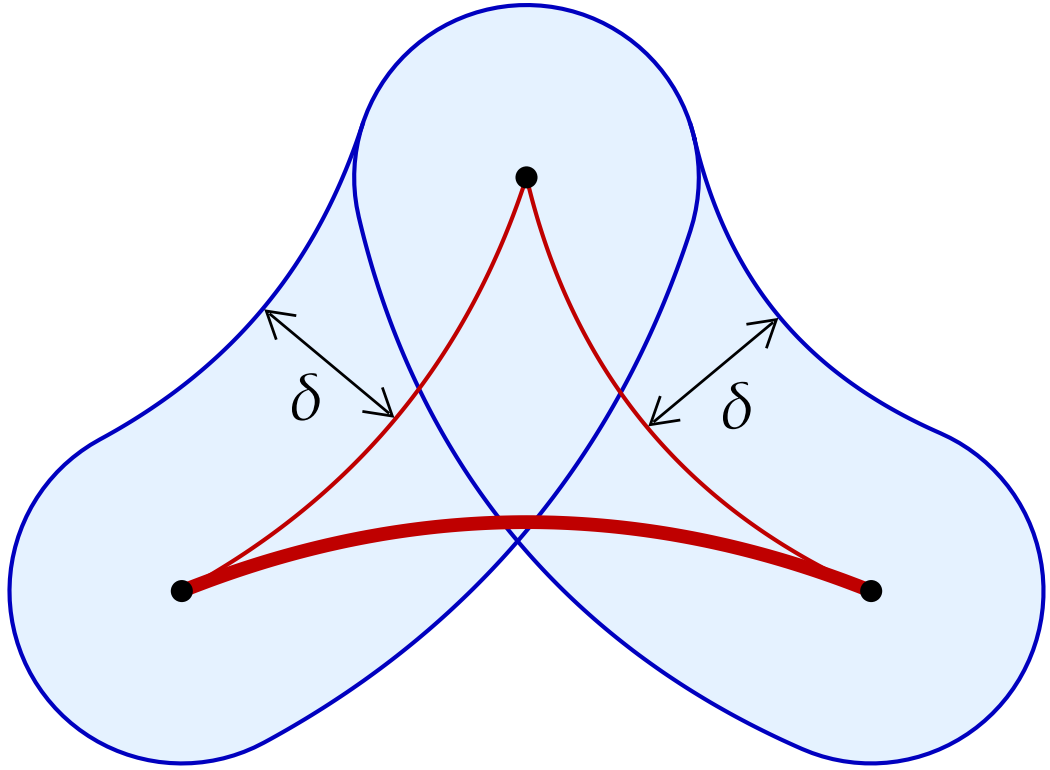
Hyperbolic groups

A locally finite graph X is **hyperbolic** if there exists $\delta \geq 0$ such that



Hyperbolic groups

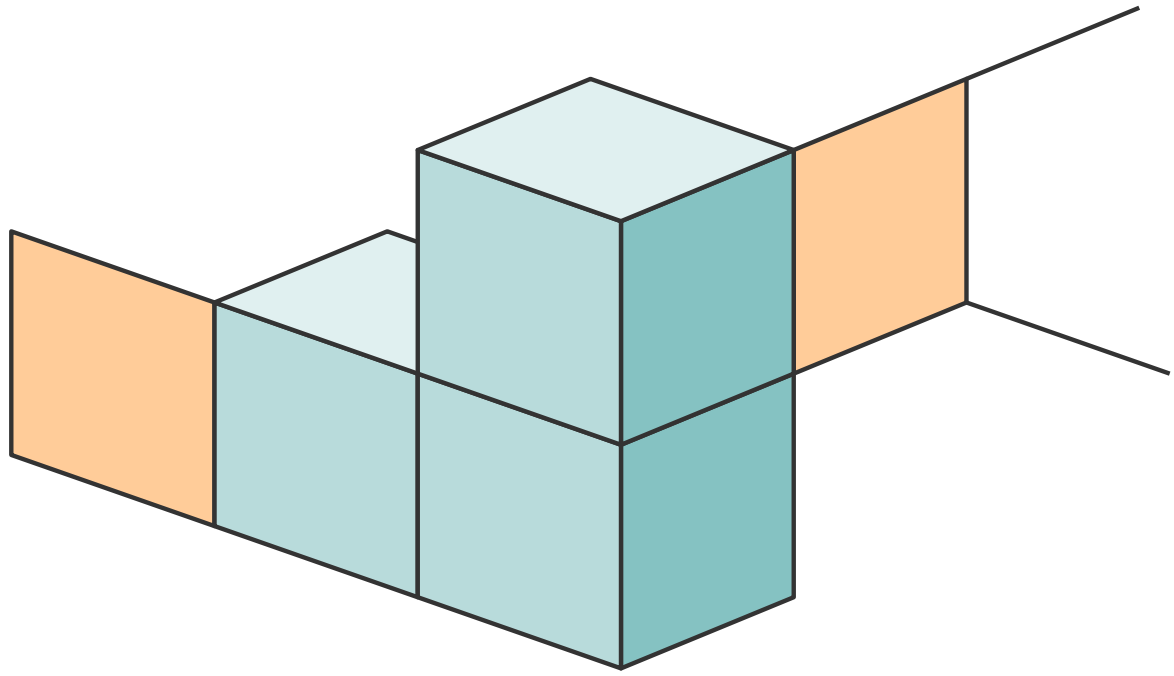
A locally finite graph X is **hyperbolic** if there exists $\delta \geq 0$ such that



holds for every geodesic triangle.

CAT(0) cubulated groups

A **cube complex** is a space made by gluing unit cubes (of different but bounded dimensions) along their isometric faces.



$\mathrm{CAT}(0)$ cubulated groups

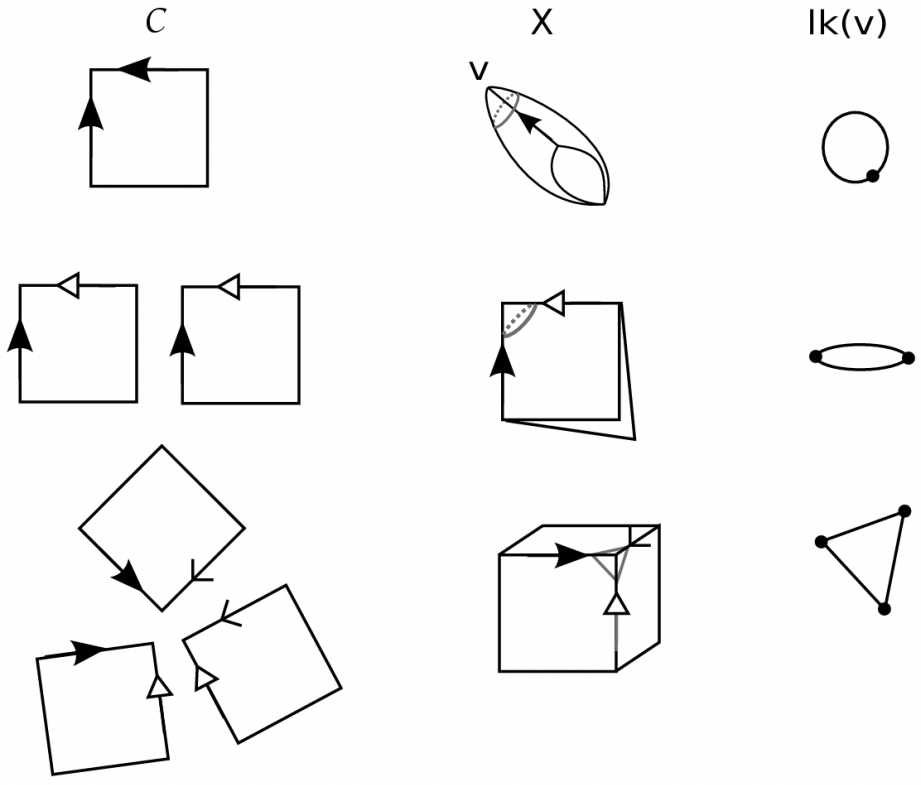
A **cube complex** is a space made by gluing unit cubes (of different but bounded dimensions) along their isometric faces.

A cube complex is said to be $\mathrm{CAT}(0)$ if it is simply connected and *the link* $\mathrm{lk}(v)$ *of every vertex* v *is a flag complex*.

$\mathrm{CAT}(0)$ cubulated groups

A **cube complex** is a space made by gluing unit cubes (of different but bounded dimensions) along their isometric faces.

A cube complex is said to be $\mathrm{CAT}(0)$ if it is simply connected and *the link* $\mathrm{lk}(v)$ *of every vertex* v *is a flag complex*.



Examples

$\mathrm{CAT}(0)$ cubulated examples:

- Surface groups
- Right Angled Artin groups
- Right Angled Coxeter groups
- Burger-Mozes f.p. simple groups

Hyperbolic examples:

- Free groups
- Fuchsian groups (discrete subgroups of $PSL_2(\mathbb{R})$)
- $\pi_1(M)$ with M compact hyperbolic manifold

Examples

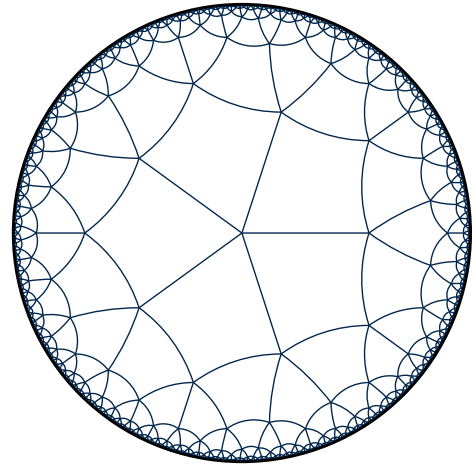
$\mathrm{CAT}(0)$ cubulated examples:

- Surface groups
- Right Angled Artin groups
- Right Angled Coxeter groups
- Burger-Mozes f.p. simple groups

Hyperbolic examples:

- Free groups
- Fuchsian groups (discrete subgroups of $PSL_2(\mathbb{R})$)
- $\pi_1(M)$ with M compact hyperbolic manifold

The Fuchsian group
$\langle r, s \mid r^5, (rs)^4 \rangle$
acting on



Examples

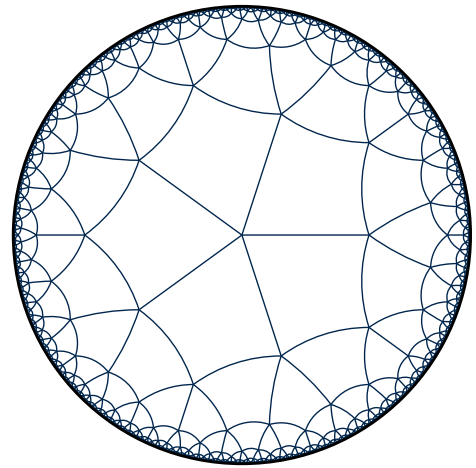
$\mathrm{CAT}(0)$ cubulated examples:

- Surface groups
- Right Angled Artin groups
- Right Angled Coxeter groups
- Burger-Mozes f.p. simple groups

Hyperbolic examples:

- Free groups
- Fuchsian groups (discrete subgroups of $PSL_2(\mathbb{R})$)
- $\pi_1(M)$ with M compact hyperbolic manifold

The Fuchsian group
$\langle r, s \mid r^5, (rs)^4 \rangle$
acting on



OpenAI announced a hyperbolic group that is not $\mathrm{CAT}(0)$

Horofunctions

A compactification for a metric space (X, d) coming from analysis:

Definition

For each $p \in X$ let $f_p : X \to \mathbb{R}$ be the function

$$f_p(x) = d(x, p) - d(x_0, p).$$

A map $f : X \to \mathbb{R}$ is a **horofunction** if there exists an unbounded sequence $\{p_n\} \subset X$ such that $f_{p_n} \to f$ uniformly on compact sets.

These functions are called the **horofunction boundary** $\partial_h X$.

Horofunctions

A compactification for a metric space (X, d) coming from analysis:

Definition

For each $p \in X$ let $f_p : X \to \mathbb{R}$ be the function

$$f_p(x) = d(x, p) - d(x_0, p).$$

A map $f : X \to \mathbb{R}$ is a **horofunction** if there exists an unbounded sequence $\{p_n\} \subset X$ such that $f_{p_n} \to f$ uniformly on compact sets.

These functions are called the **horofunction boundary** $\partial_h X$.

For a group G finitely generated by a subset S, the horofunction boundary $\partial_h X$ of its Cayley graph X is a metrizable Stone space $\Longrightarrow \partial_h X \subseteq A^\omega$ with A finite alphabet.

...and other boundaries

...and other boundaries

Hyperbolic case:
Hyperbolic groups have another natural boundary ∂X called the *Gromov boundary* given by equivalence classes of semi-infinite rays which eventually travel at a bounded distance. There is a finite-to-one projection $\partial_h X \twoheadrightarrow \partial X$ with ∂X.

...and other boundaries

Hyperbolic case:
Hyperbolic groups have another natural boundary ∂X called the *Gromov boundary* given by equivalence classes of semi-infinite rays which eventually travel at a bounded distance. There is a finite-to-one projection $\partial_h X \twoheadrightarrow \partial X$ with ∂X.

CCC case:
$\partial_h X^{(1)} \simeq \partial X$ with ∂X the Roller boundary and $X^{(1)}$ the 1-skeleton of the $\mathrm{CAT}(0)$ cubical complex.

Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^* the collection of all finite words on X.

Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^ω the collection of all infinite words on X.

Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^ω the collection of all infinite words on X.

A **language** L is a subset of X^* or X^ω.

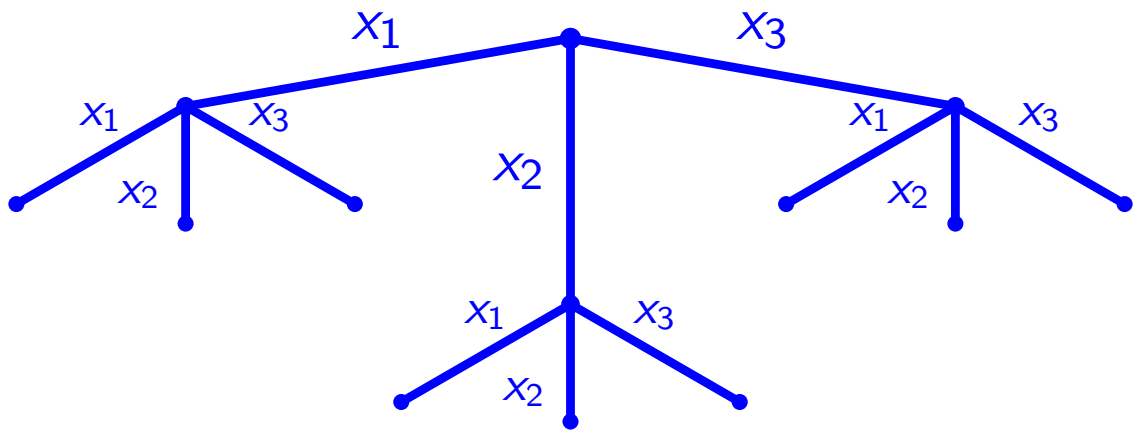
Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^ω the collection of all infinite words on X.

A **language** L is a subset of X^* or X^ω.

Graphical interpretation: X^* is a rooted tree with boundary X^ω.



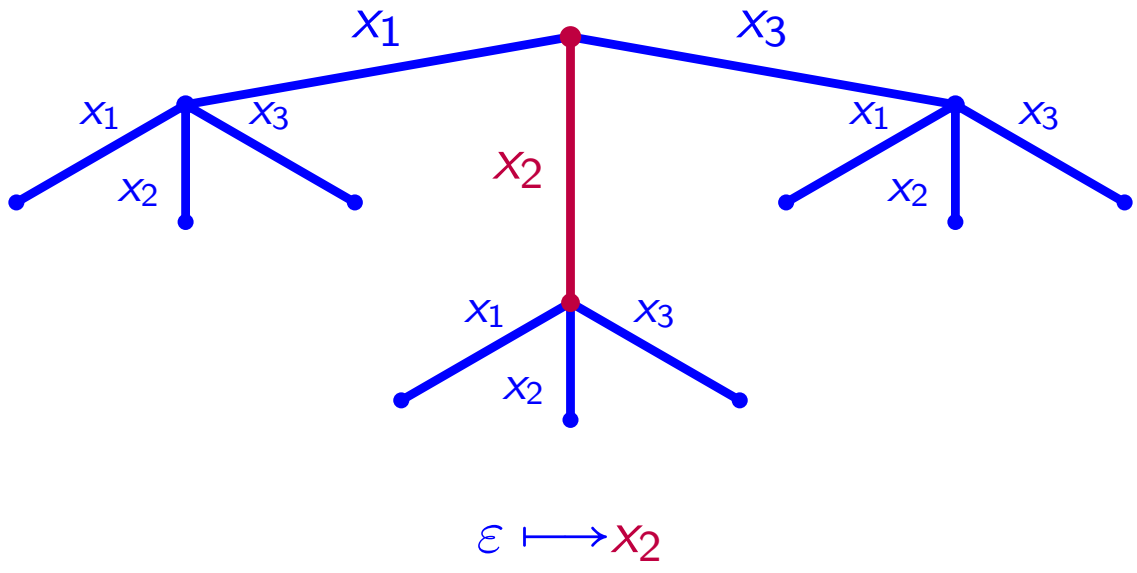
Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^ω the collection of all infinite words on X.

A **language** L is a subset of X^* or X^ω.

Graphical interpretation: X^* is a rooted tree with boundary X^ω.



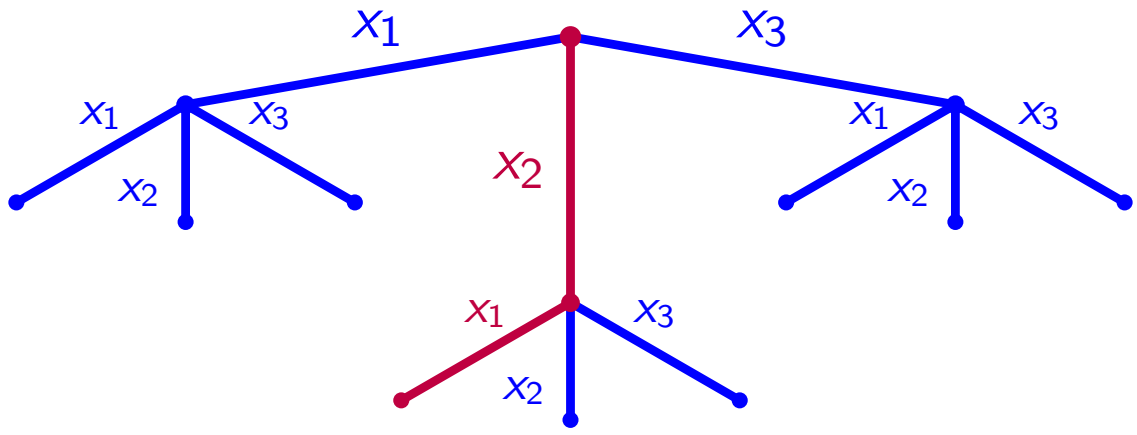
Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^ω the collection of all infinite words on X.

A **language** L is a subset of X^* or X^ω.

Graphical interpretation: X^* is a rooted tree with boundary X^ω.



$\varepsilon \longmapsto x_2 \longmapsto x_2 x_1$

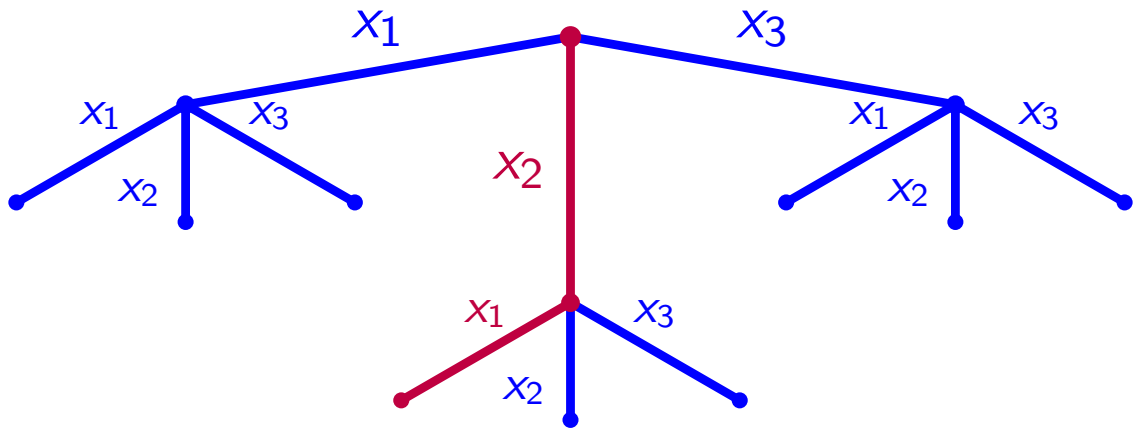
Formal languages and trees

An **alphabet** is just a finite set $X = \{x_1, x_2, \ldots x_n\}$.

We denote by X^ω the collection of all infinite words on X.

A **language** L is a subset of X^* or X^ω.

Graphical interpretation: X^* is a rooted tree with boundary X^ω.



$$\varepsilon \longmapsto x_2 \longmapsto x_2x_1 \longmapsto x_2x_1\ldots$$

Normal form boundary

Let $G = \langle X \rangle$ be a group with X finite and symmetric.

A **normal form** is a language $L \subset X^*$ such that the surjection $\pi : X^* \twoheadrightarrow G$ is bijective when restricted to L.

Normal form boundary

Let $G = \langle X \rangle$ be a group with X finite and symmetric.

A **normal form** is a language $L \subset X^*$ such that the surjection $\pi : X^* \twoheadrightarrow G$ is bijective when restricted to L.

The **boundary** $\partial L \subseteq X^\omega$ of L is the set of accumulation points of $L \subseteq X^*$, where $X^* \cup X^\omega$ has the (ultra)metric given by measuring 2^{-n}, where n is the length of the greatest common prefix.

Normal form boundary

Let $G = \langle X \rangle$ be a group with X finite and symmetric.

A **normal form** is a language $L \subset X^*$ such that the surjection $\pi : X^* \twoheadrightarrow G$ is bijective when restricted to L.

The **boundary** $\partial L \subseteq X^\omega$ of L is the set of accumulation points of $L \subseteq X^*$, where $X^* \cup X^\omega$ has the (ultra)metric given by measuring 2^{-n}, where n is the length of the greatest common prefix.

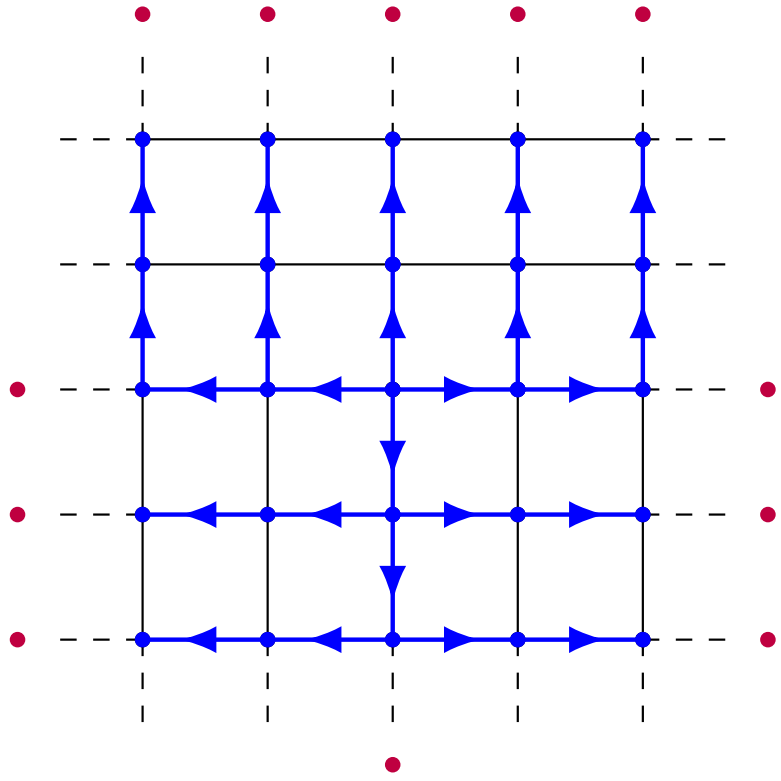
We say that L is a continuous normal form if the natural action of G on L

$$g \cdot w = \pi^{-1}(g \cdot (\pi w)g)$$

extends continuously on $L \cup \partial L$. In this case, ∂L is the normal form boundary.

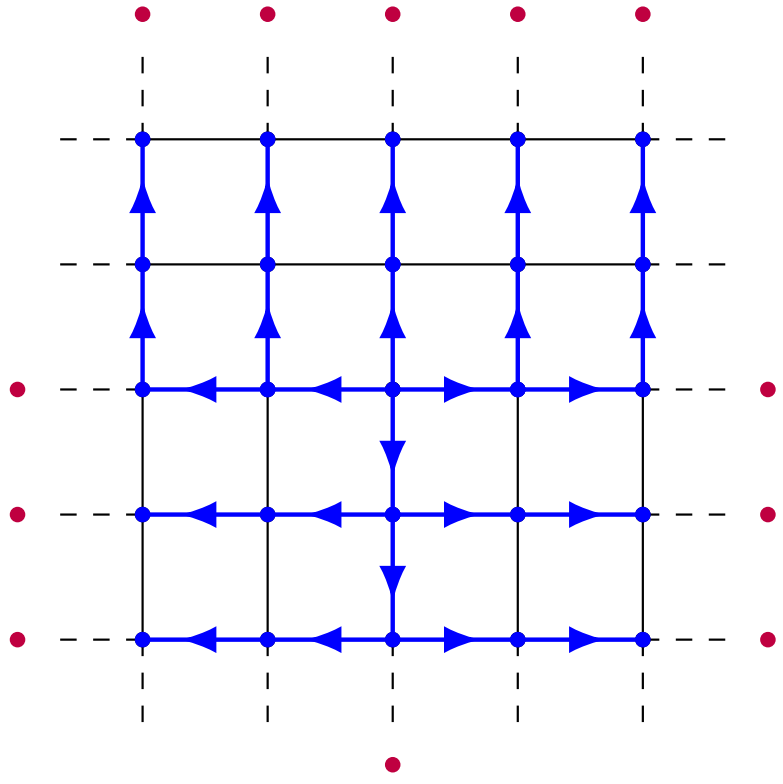
Interesting (non-)examples

Consider $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$.



Interesting (non-)examples

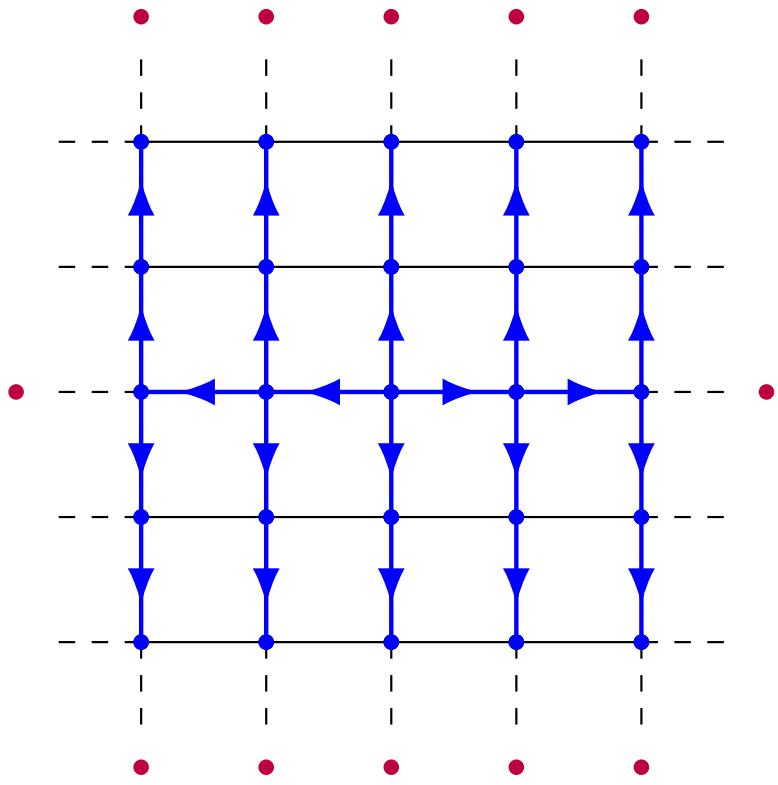
Consider $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$.



NOT CONTINUOUS

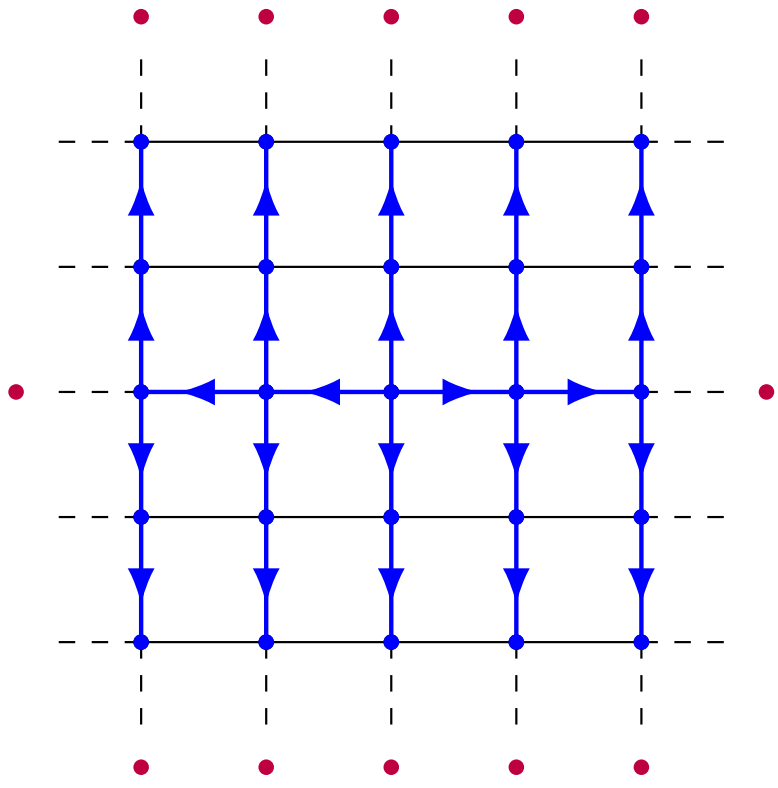
Interesting (non-)examples

Consider $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$.



Interesting (non-)examples

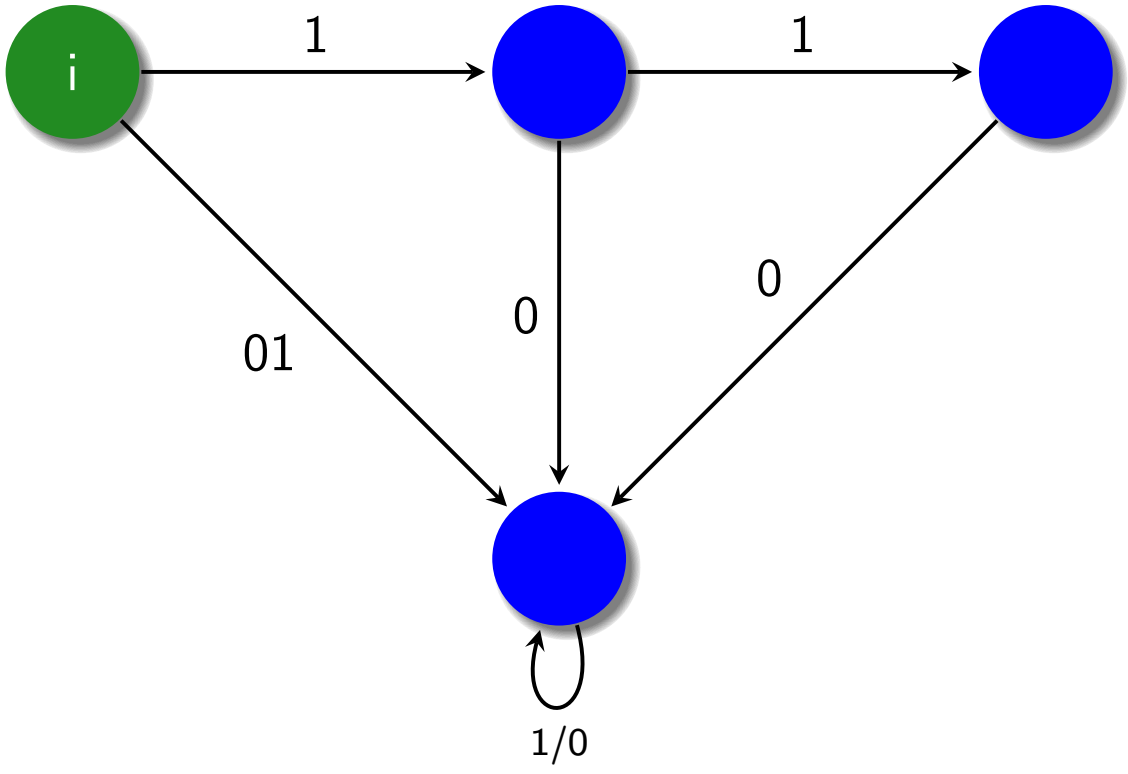
Consider $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$.



CONTINUOUS, NOT FAITHFUL

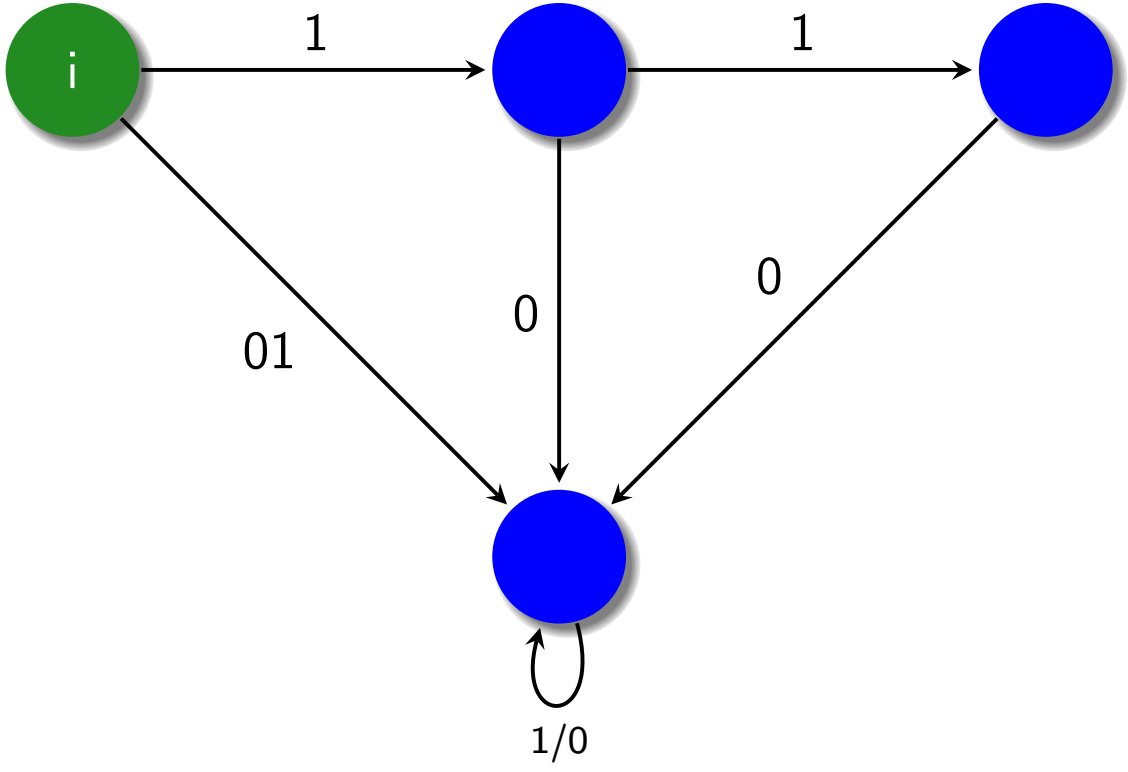
Finite state automata

Input: finite or infinite word **Output**: YES/NO



Finite state automata

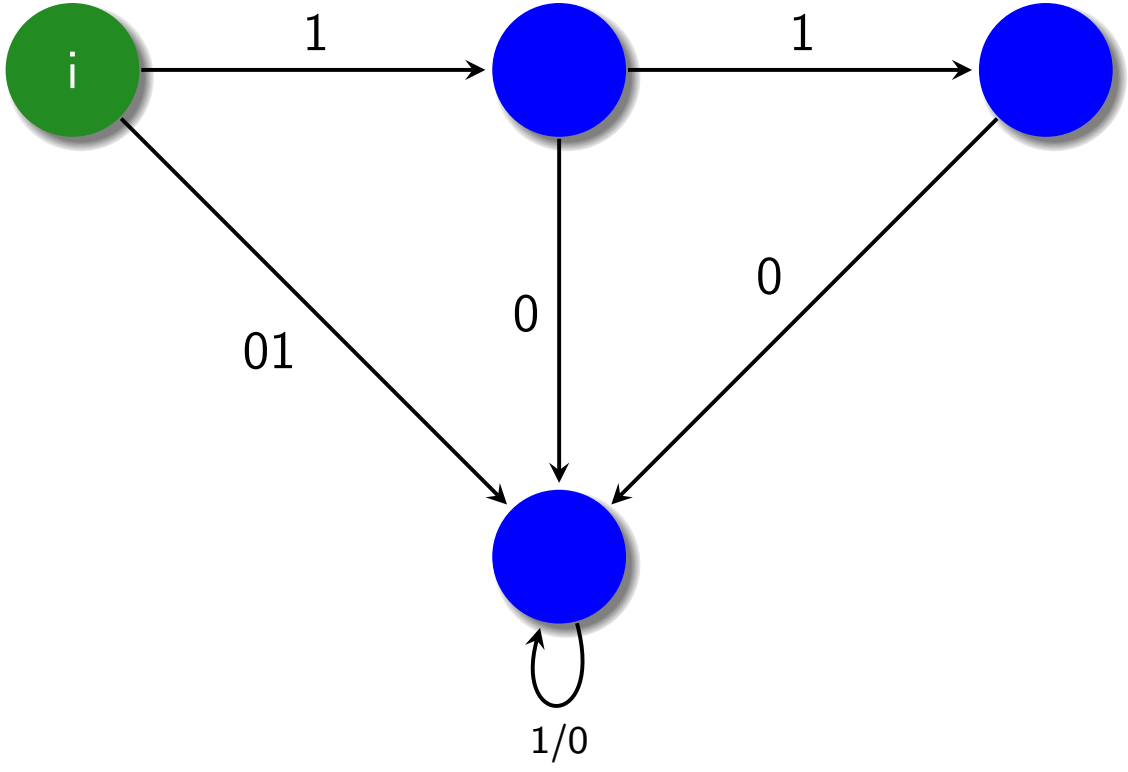
Input: finite or infinite word **Output**: YES/NO



110 . . . 111 . . .

Finite state automata

Input: finite or infinite word **Output**: YES/NO

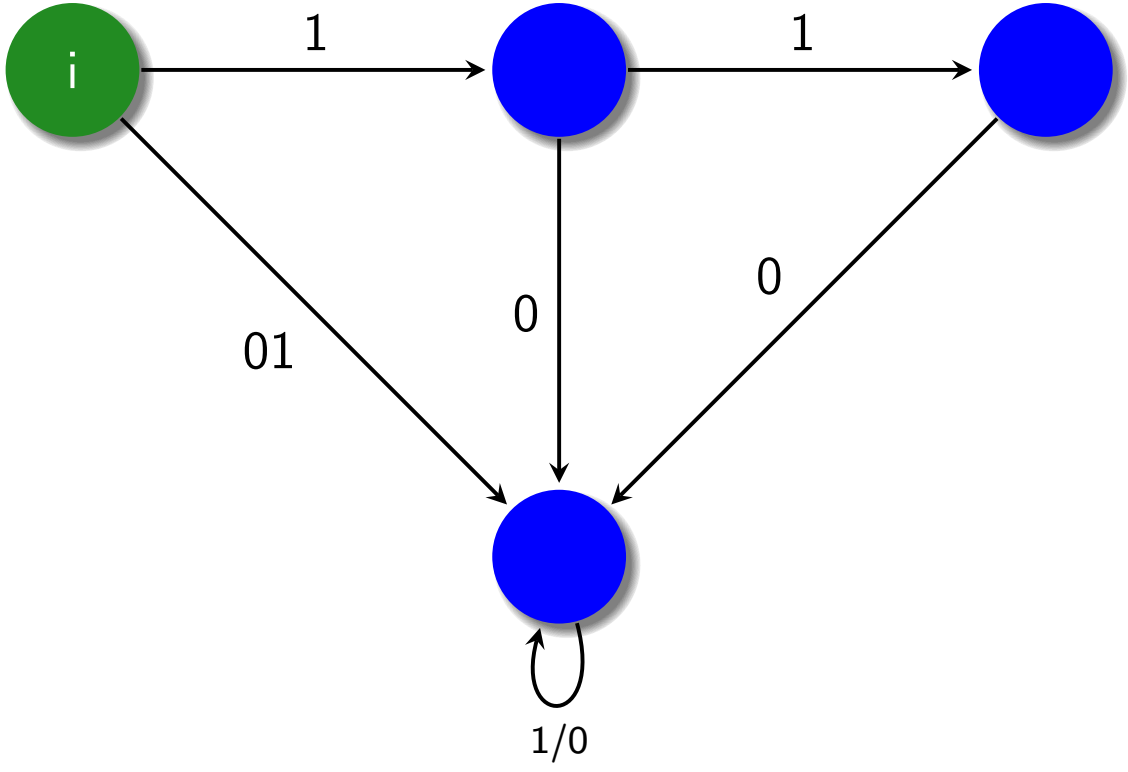


110...

111...

Finite state automata

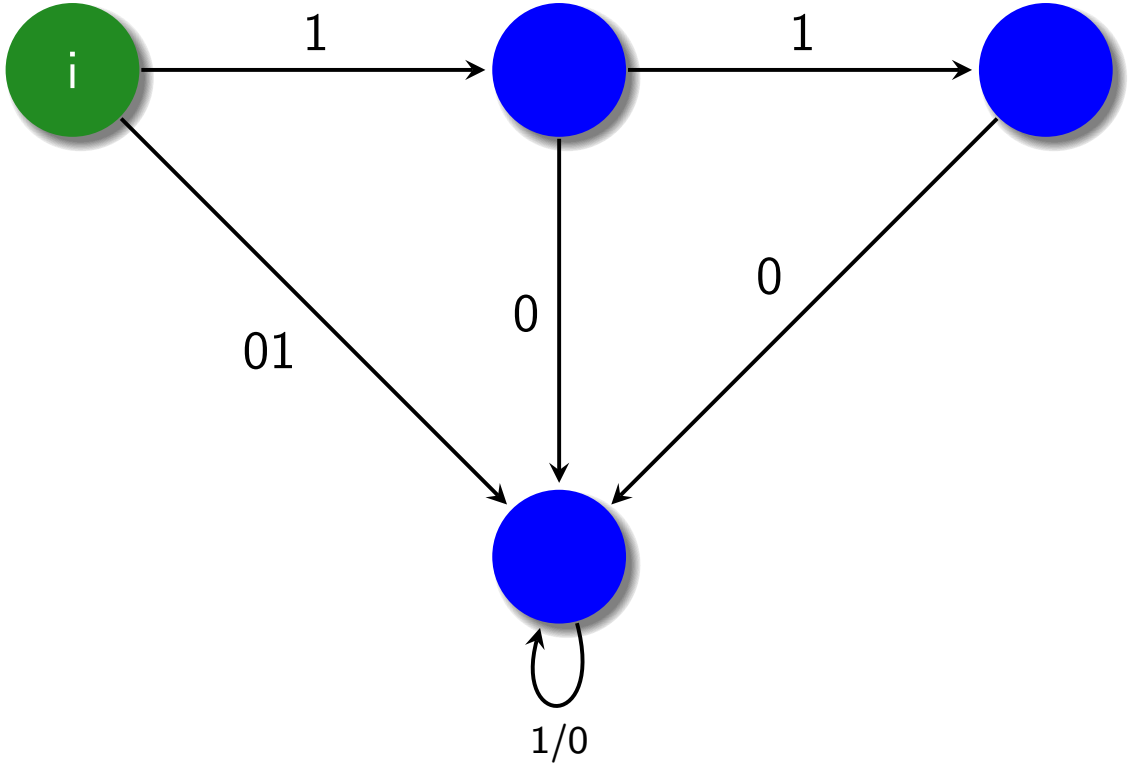
Input: finite or infinite word **Output**: YES/NO



110 . . . 111 . . .

Finite state automata

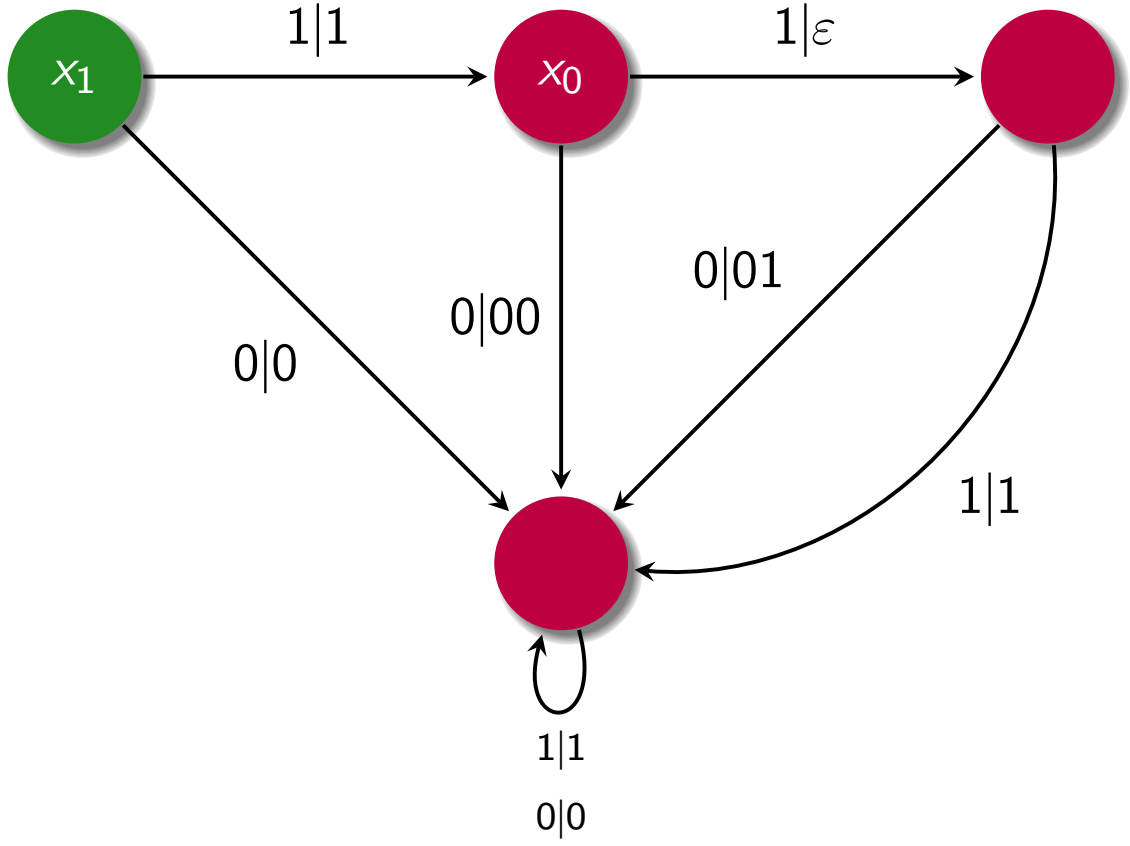
Input: finite or infinite word **Output**: YES/NO



$110\ldots \rightsquigarrow$ YES $111\ldots \rightsquigarrow$ NO

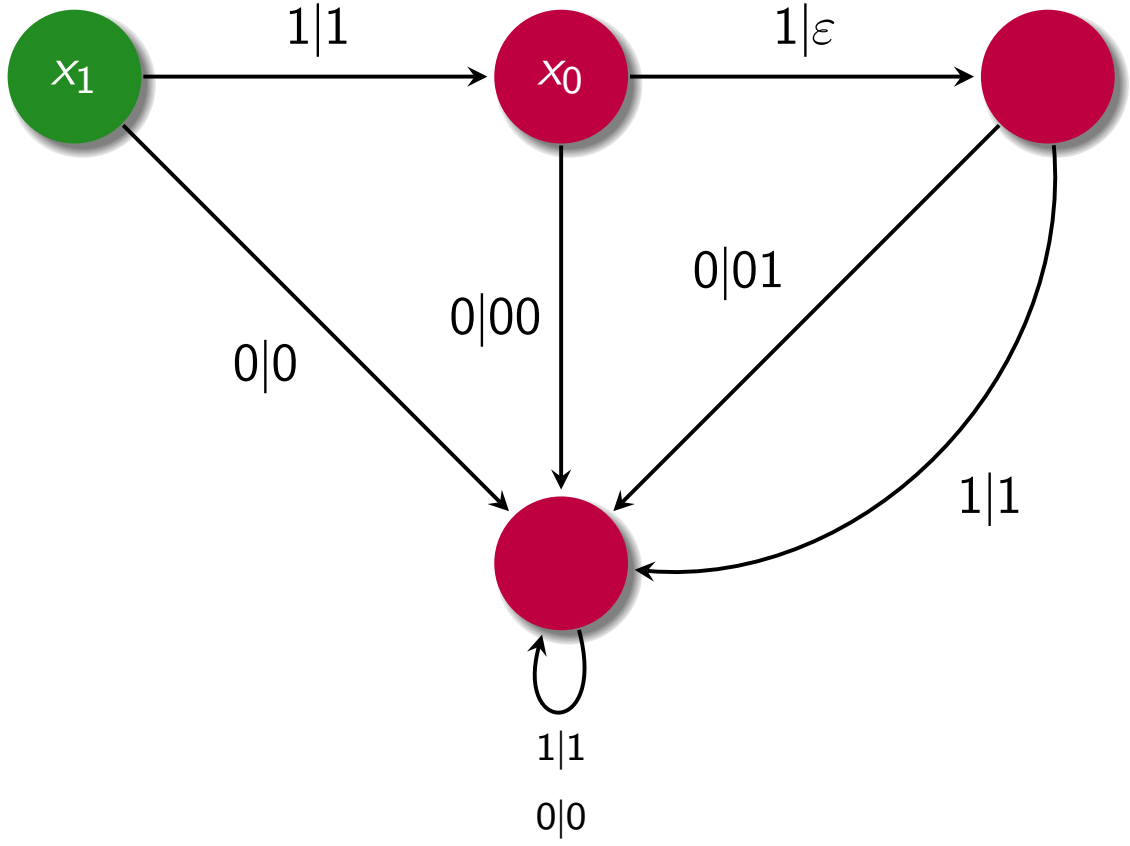
Finite state transducers

Input: infinite word **Output**: infinite word



Finite state transducers

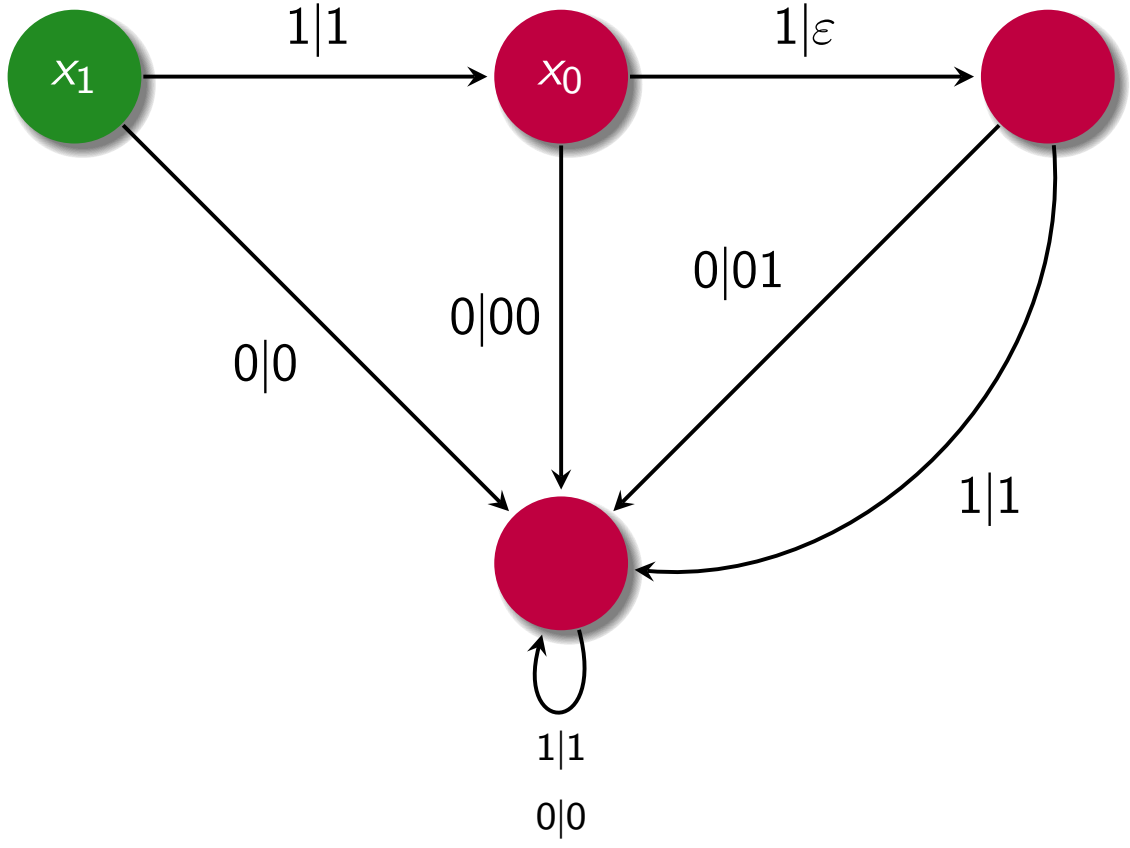
Input: infinite word **Output**: infinite word



11001...

Finite state transducers

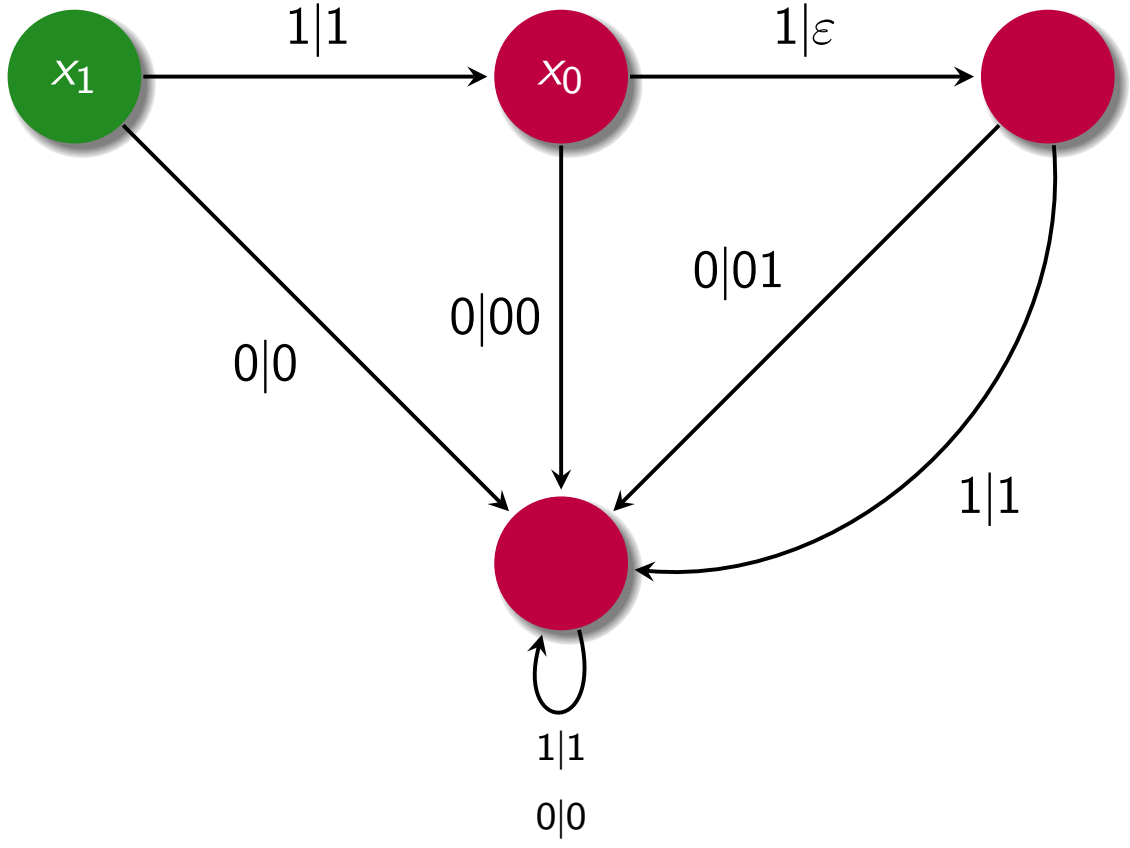
Input: infinite word **Output**: infinite word



$11001\ldots \xrightarrow{x_1} 11001\ldots$

Finite state transducers

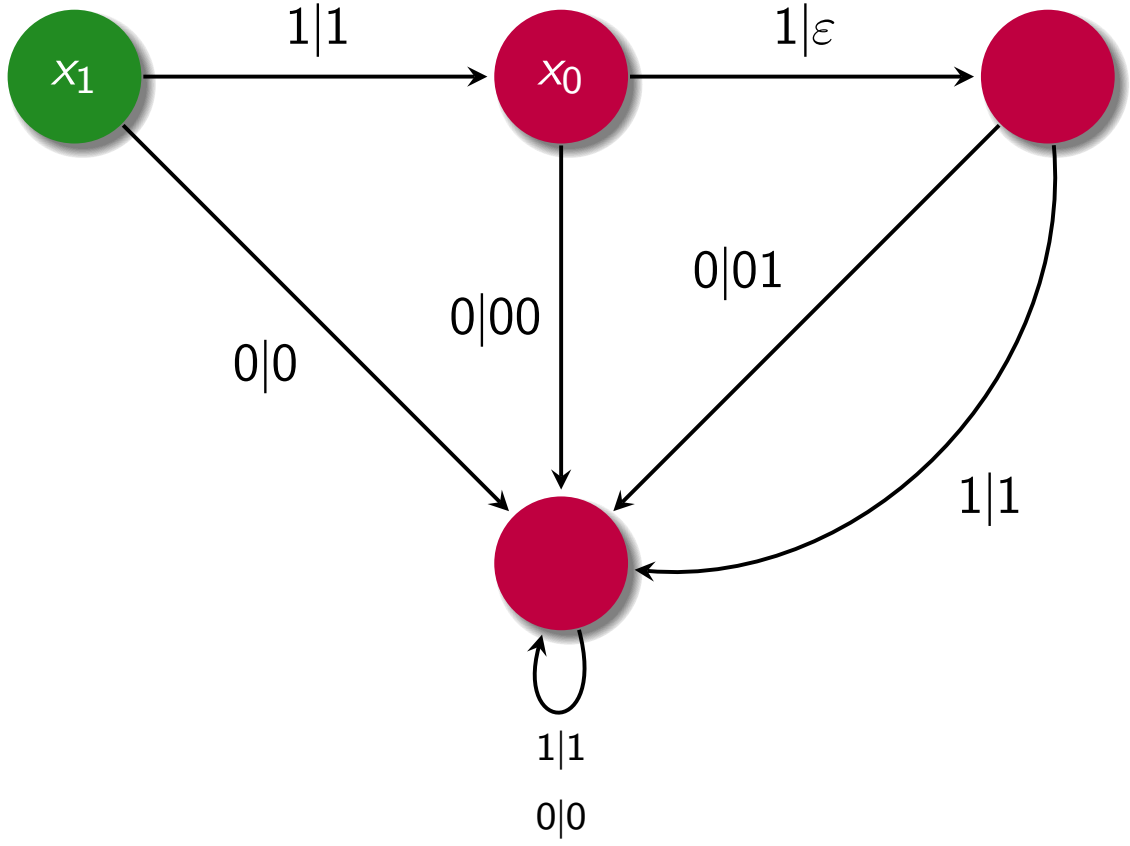
Input: infinite word **Output**: infinite word



$$11001\ldots \xrightarrow{x_1} 11001\ldots \xrightarrow{x_1} 1001\ldots$$

Finite state transducers

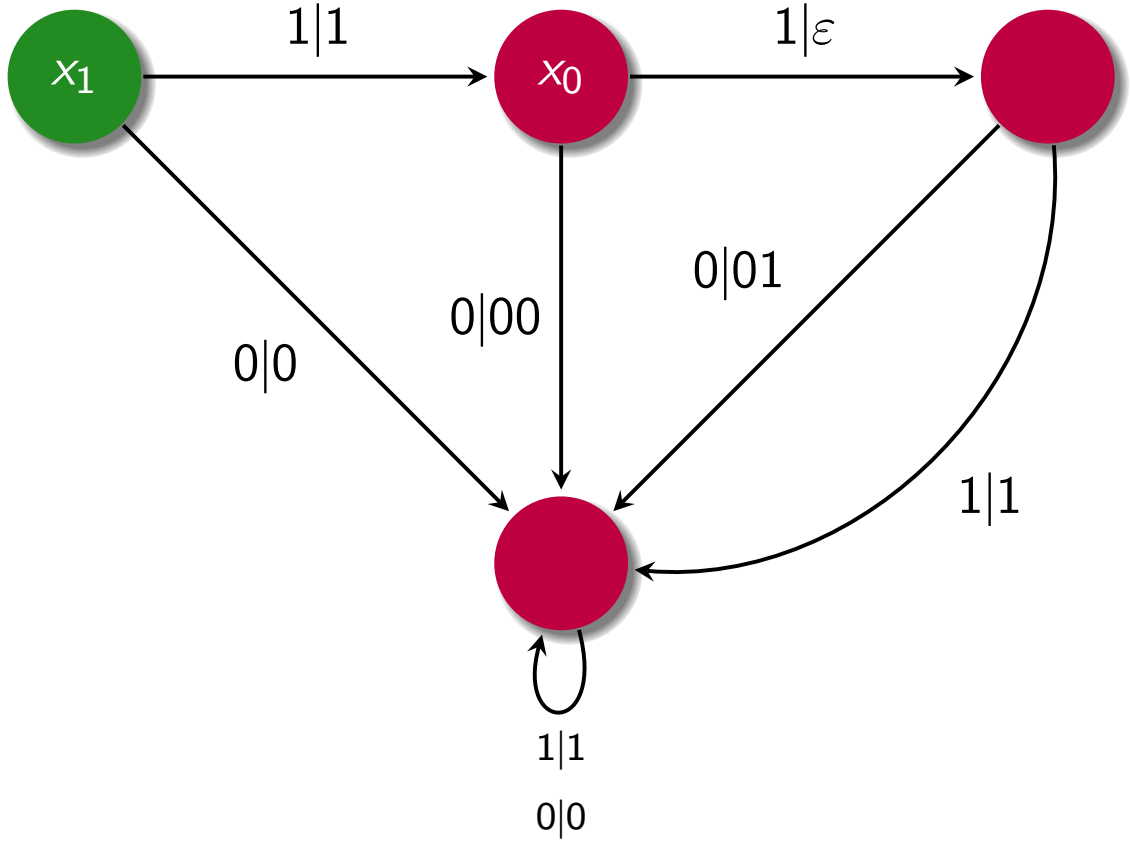
Input: infinite word **Output**: infinite word



$$11001\ldots \xmapsto{x_1} 11001\ldots \xmapsto{x_1} 1001\ldots \xmapsto{x_1} 10101\ldots$$

Finite state transducers

Input: infinite word **Output**: infinite word



$$11001\ldots \xrightarrow{x_1} 11001\ldots \xrightarrow{x_1} 1001\ldots \xrightarrow{x_1} 10101\ldots \xrightarrow{x_1} 10101\ldots$$

Groups and machines

Epstein, Cannon, Holt, Levy, Peterson, Thurston, 1992: A group G is **automatic** if there exist(s)

1. a normal form L accepted by a FSA M_ε;
2. a set $\{M_x : x \in X\}$ of FSA where M_x accepts $(w_1, w_2) \in L^2$ precisely when $x \cdot w_1 = w_2$ in G.

Groups and machines

Epstein, Cannon, Holt, Levy, Peterson, Thurston, 1992: A group G is **automatic** if there exist(s)

1. a normal form L accepted by a FSA M_ε;
2. a set $\{M_x : x \in X\}$ of FSA where M_x accepts $(w_1, w_2) \in L^2$ precisely when $x \cdot w_1 = w_2$ in G.

Grigorchuk, Nekrashevych, Suschanskii, 2000: for a finite alphabet A, the rational group $\mathcal{R}_A$ is the group of homeomorphisms of A^ω that can be represented by invertible FSTs.

Groups and machines

Epstein, Cannon, Holt, Levy, Peterson, Thurston, 1992: A group G is **automatic** if there exist(s)

1. a normal form L accepted by a FSA M_ε;
2. a set $\{M_x : x \in X\}$ of FSA where M_x accepts $(w_1, w_2) \in L^2$ precisely when $x \cdot w_1 = w_2$ in G.

Grigorchuk, Nekrashevych, Suschanskii, 2000: for a finite alphabet A, the rational group $\mathcal{R}_A$ is the group of homeomorphisms of A^ω that can be represented by invertible FSTs.

Note that any two rational groups $\mathcal{R}_A$ and $\mathcal{R}_B$ over two finite alphabets are isomorphic. We can consider the **rational group** $\mathcal{R}$.

Groups and machines

Epstein, Cannon, Holt, Levy, Peterson, Thurston, 1992: A group G is **automatic** if there exist(s)

1. a normal form L accepted by a FSA M_ε;
2. a set $\{M_x : x \in X\}$ of FSA where M_x accepts $(w_1, w_2) \in L^2$ precisely when $x \cdot w_1 = w_2$ in G.

Grigorchuk, Nekrashevych, Suschanskii, 2000: for a finite alphabet A, the rational group $\mathcal{R}_A$ is the group of homeomorphisms of A^ω that can be represented by invertible FSTs.

Note that any two rational groups $\mathcal{R}_A$ and $\mathcal{R}_B$ over two finite alphabets are isomorphic. We can consider the **rational group** $\mathcal{R}$.

$\mathcal{R}$ is simple and infinitely generated (Belk-Hyde-M, 2019)

Examples

Automatic groups:

- Hyperbolic groups (Cannon, 1992)
- $\mathrm{CAT}(0)$ cubulated groups (Niblo-Reeves, 1998)
- Families of Artin groups (e.g. spherical type, Charney, 1992)
- Baumslag-Solitar groups (Thurston, 1992)

Examples

Automatic groups:

- Hyperbolic groups (Cannon, 1992)
- $\mathrm{CAT}(0)$ cubulated groups (Niblo-Reeves, 1998)
- Families of Artin groups (e.g. spherical type, Charney, 1992)
- Baumslag-Solitar groups (Thurston, 1992)

Rational groups:

- Thompson groups F, T, V and generalizations
- Self-similar groups
- Hyperbolic groups (Belk-Bleak-M, 2021)

Examples

Automatic groups:

- Hyperbolic groups (Cannon, 1992)
- $\mathrm{CAT}(0)$ cubulated groups (Niblo-Reeves, 1998)
- Families of Artin groups (e.g. spherical type, Charney, 1992)
- Baumslag-Solitar groups (Thurston, 1992)

Rational groups:

- Thompson groups F, T, V and generalizations
- Self-similar groups
- Hyperbolic groups (Belk-Bleak-M, 2021)

Question: Is there any interaction between the two classes?

Main theorem

A group which is automatic with respect to a continuous normal form is called **continuous automatic**.

Theorem

If G is a continuous automatic group, then G acts rationally on ∂L and hence embeds in the rational group $\mathcal{R}$.

Main theorem

A group which is automatic with respect to a continuous normal form is called **continuous automatic**.

Theorem

If G is a continuous automatic group, then G acts rationally on ∂L and hence embeds in the rational group $\mathcal{R}$.

Corollary

The following groups are continuous automatic (hence rational):

1. *Artin groups of spherical type;*
2. *Baumslag-Solitar groups $B(n, m)$ (asynchronous continuous automatic);*
3. *Torsion-free $\mathrm{CAT}(0)$ cubulated groups.*

Continuity criterion

Proposition

Let G be a group, let X be a semigroup generating set for G, and let $L \subseteq X^*$ be a language of normal forms for G.

Then L is continuous if and only if for every semigroup generating set Y, every $y \in Y$ and $N \in \mathbb{N}$ there exists an $M \in \mathbb{N}$ so that

$$|\textit{greatest common prefix}(g,h)|_L \geq M \quad \Rightarrow$$
$$|\textit{greatest common prefix}(yg,yh)|_L \geq N$$

for all $g, h \in G$.

Closure properties

Proposition

Continuous automatic groups contain finite groups and free groups, and are closed under:

- *direct products;*
- *free products;*
- *passing to and from finite-index subgroups.*

Closure properties

Proposition

Continuous automatic groups contain finite groups and free groups, and are closed under:

- *direct products;*
- *free products;*
- *passing to and from finite-index subgroups.*

Corollary

Every finitely generated abelian group is continuous automatic.

Closure properties

Proposition

Continuous automatic groups contain finite groups and free groups, and are closed under:

- *direct products;*
- *free products;*
- *passing to and from finite-index subgroups.*

Corollary

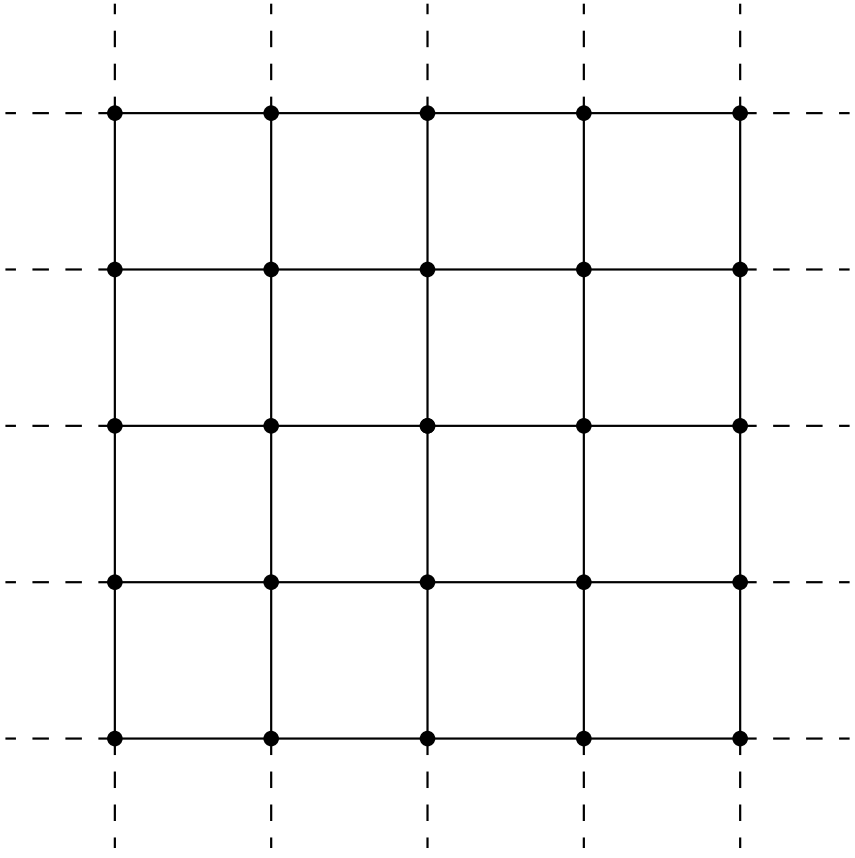
Every finitely generated abelian group is continuous automatic.

Corollary

Every continuous automatic group embeds in one whose normal-form boundary is a Cantor space and whose boundary action is faithful.

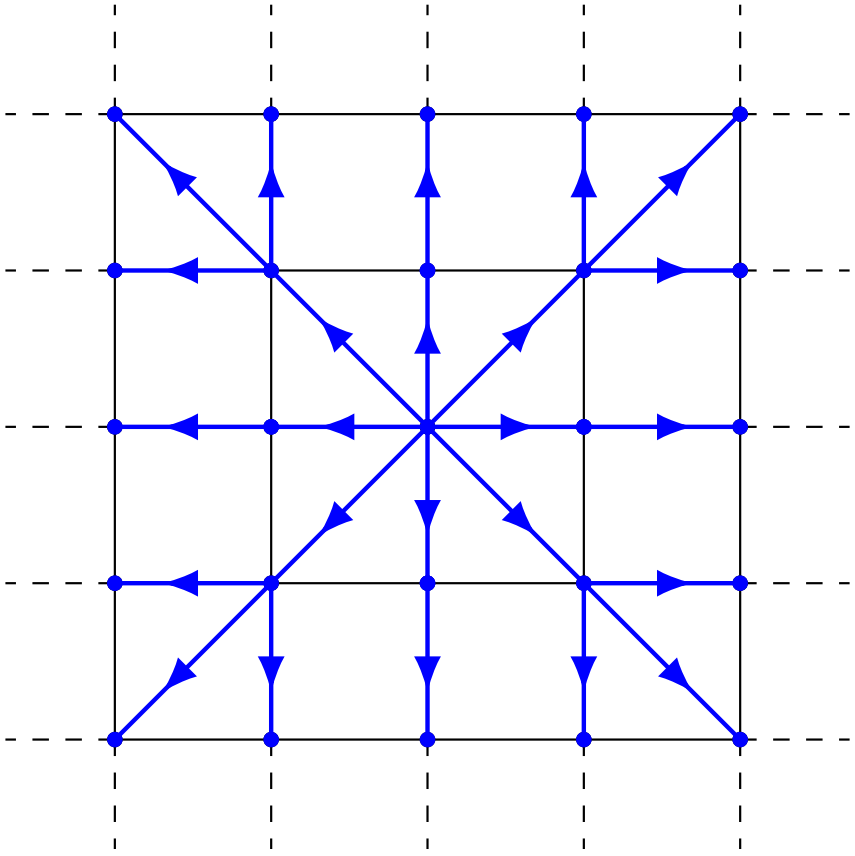
The normal form for a CCC

Consider the $\mathrm{CAT}(0)$ cubulated group $\mathbb{Z}^2$ acting on the CCC $\mathbb{R}^2$



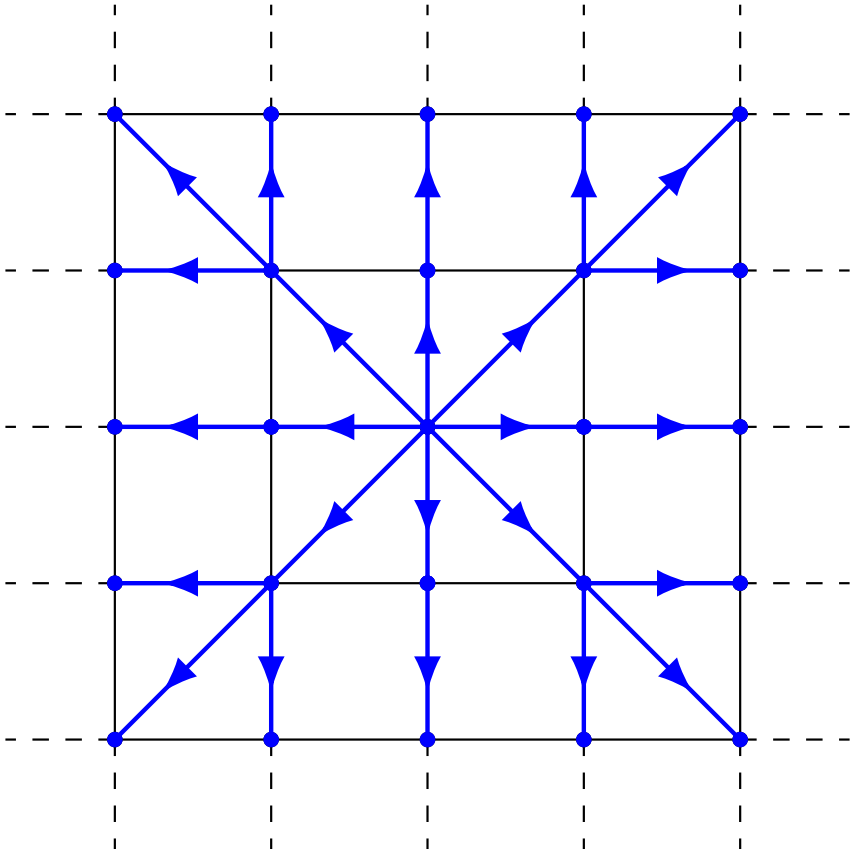
The normal form for a CCC

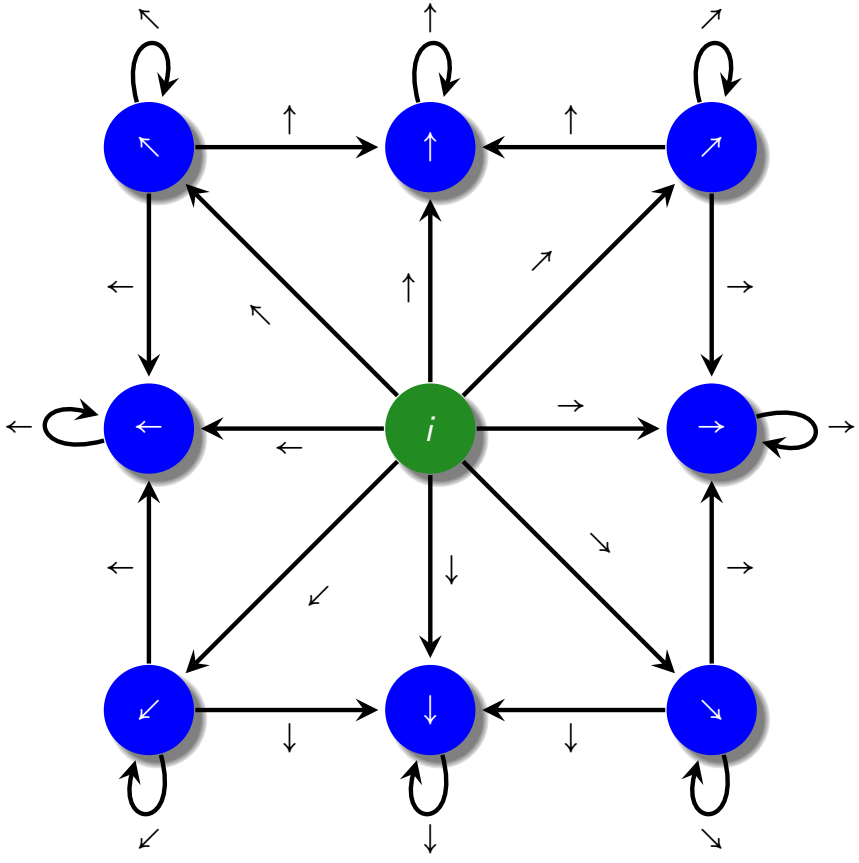
Consider the $\mathrm{CAT}(0)$ cubulated group $\mathbb{Z}^2$ acting on the CCC $\mathbb{R}^2$



The normal form for a CCC

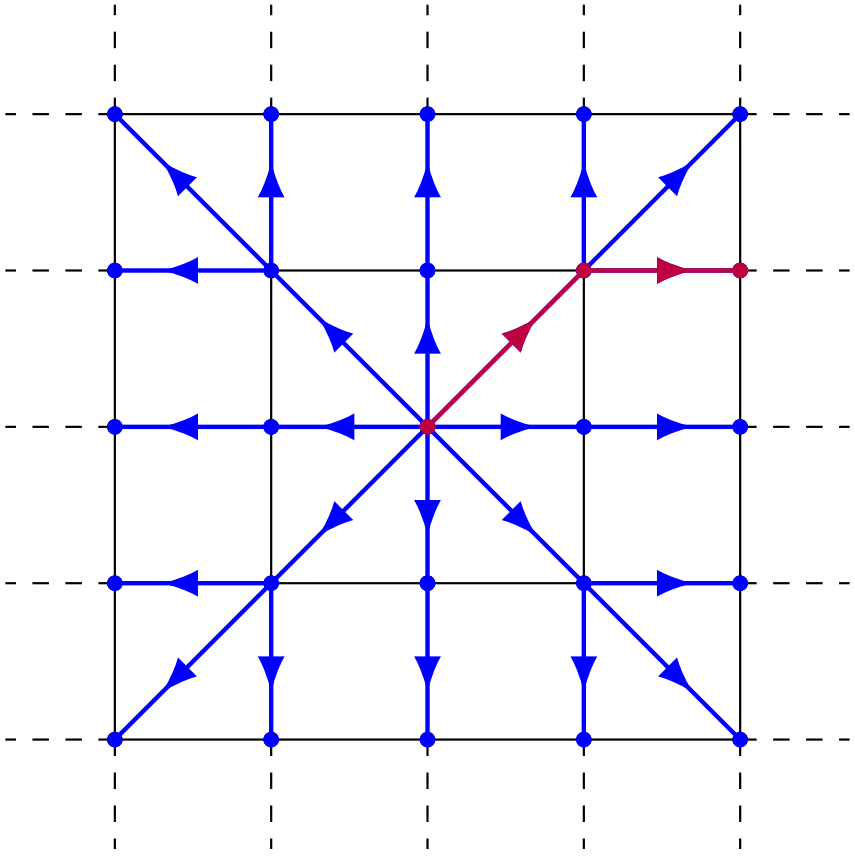
Consider the $\mathrm{CAT}(0)$ cubulated group $\mathbb{Z}^2$ acting on the CCC $\mathbb{R}^2$

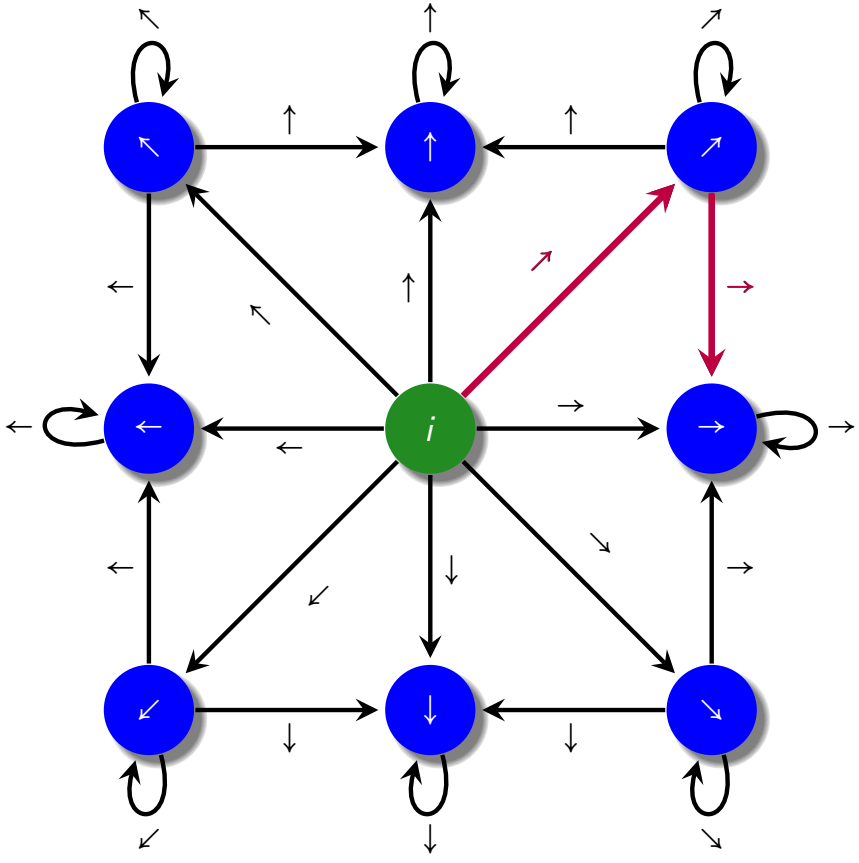




The normal form for a CCC

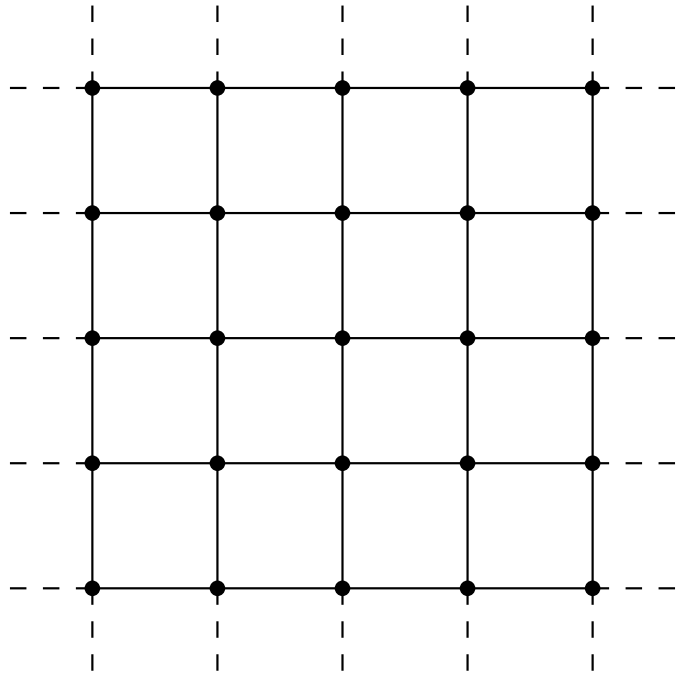
Consider the $\mathrm{CAT}(0)$ cubulated group $\mathbb{Z}^2$ acting on the CCC $\mathbb{R}^2$





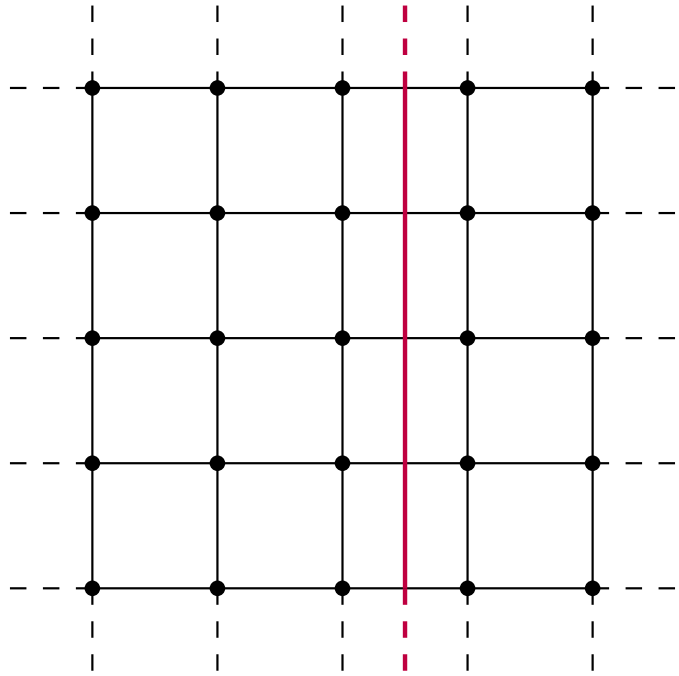
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two *half-spaces*.
An *ultrafilter* is...



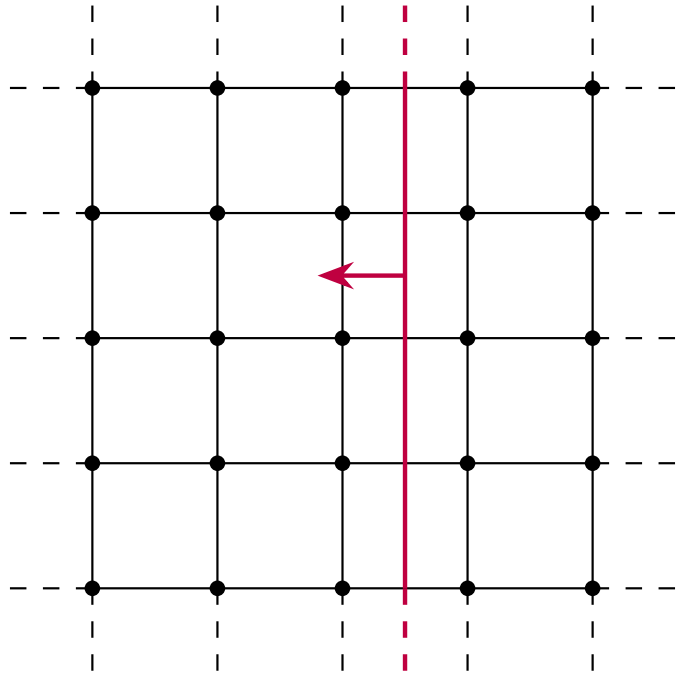
The Roller boundary

A CCC comes with a family of hyperplanes.
Each hyperplane divides the CCC into two *half-spaces*.
An *ultrafilter* is...



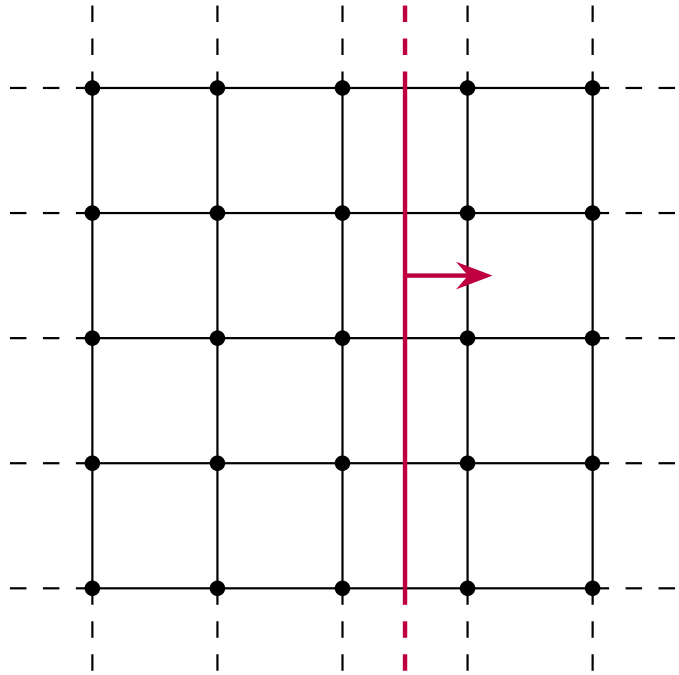
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two half-spaces.
An *ultrafilter* is...



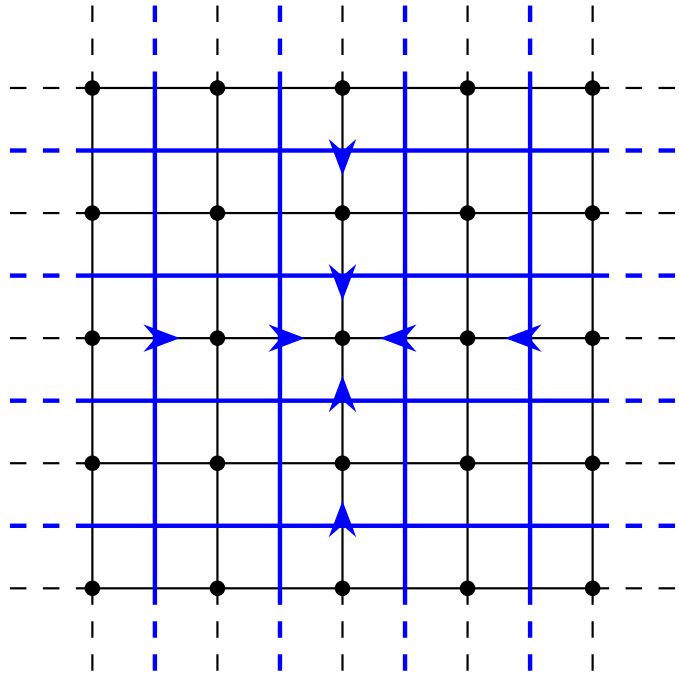
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two half-spaces.
An *ultrafilter* is...



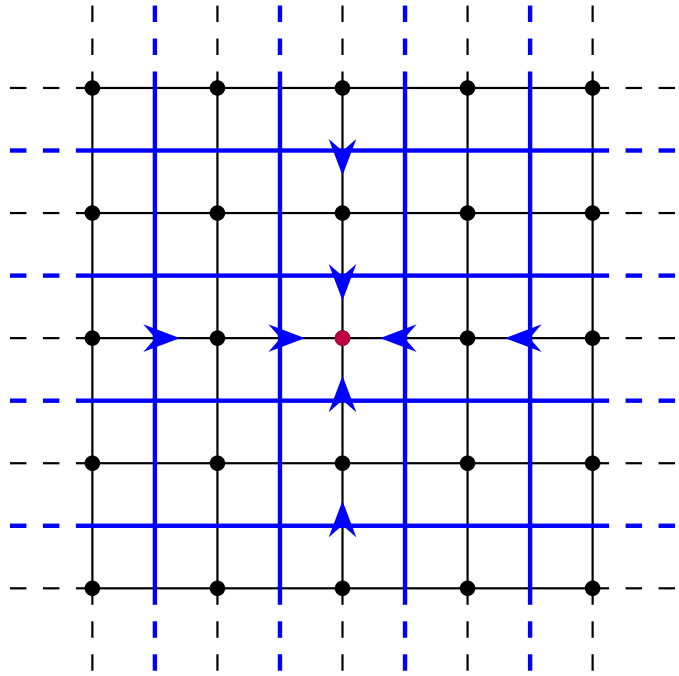
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two *half-spaces*.
An ultrafilter is...



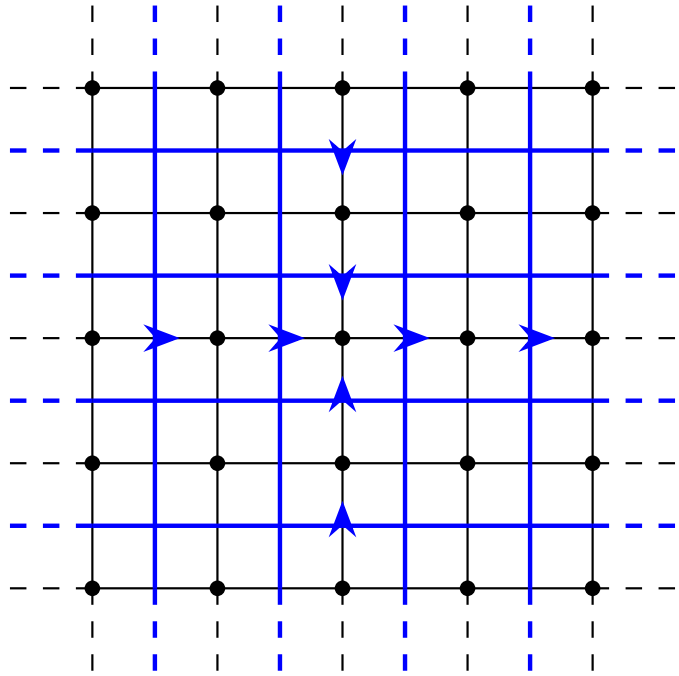
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two *half-spaces*.
An ultrafilter is...



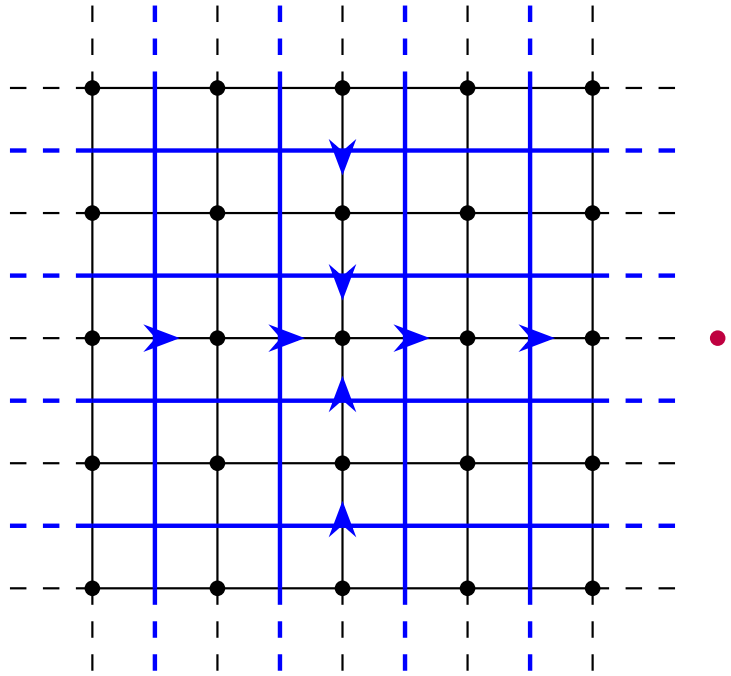
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two *half-spaces*.
An ultrafilter is...



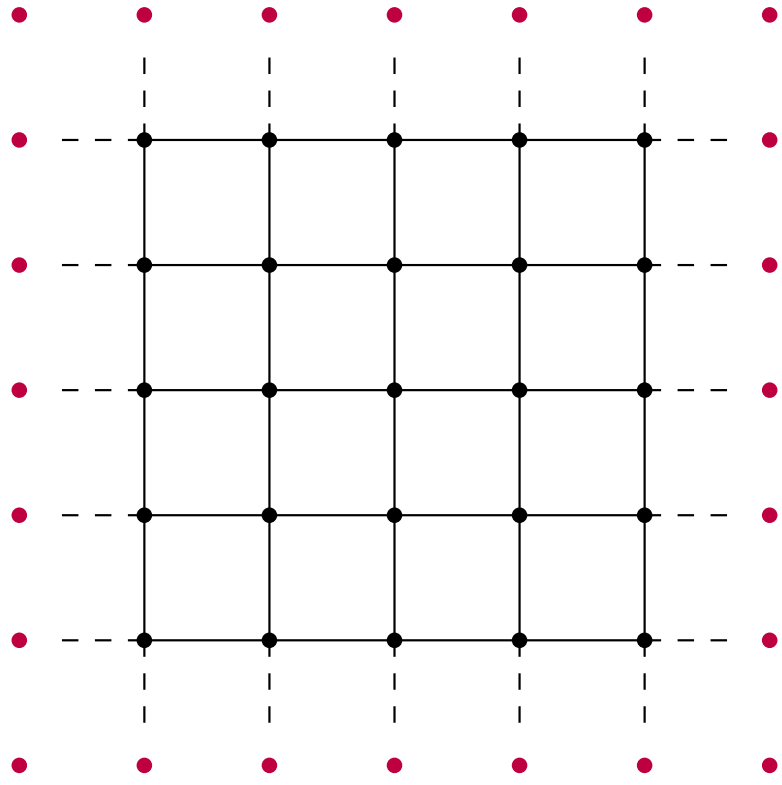
The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two *half-spaces*.
An ultrafilter is...



The Roller boundary

A CCC comes with a family of *hyperplanes*.
Each hyperplane divides the CCC into two *half-spaces*.
An *ultrafilter* is...



The Roller boundary ∂X is the collection of ultrafilters which do not represent vertices.

Equivalence of boundaries

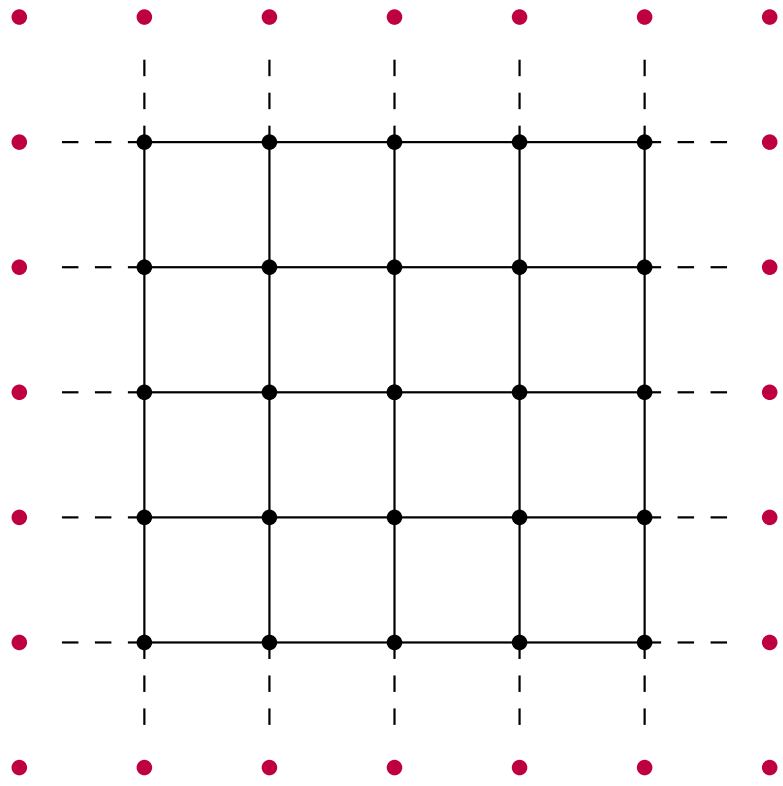
Theorem

The boundaries ∂X and ∂L are homeomorphic.

Equivalence of boundaries

Theorem

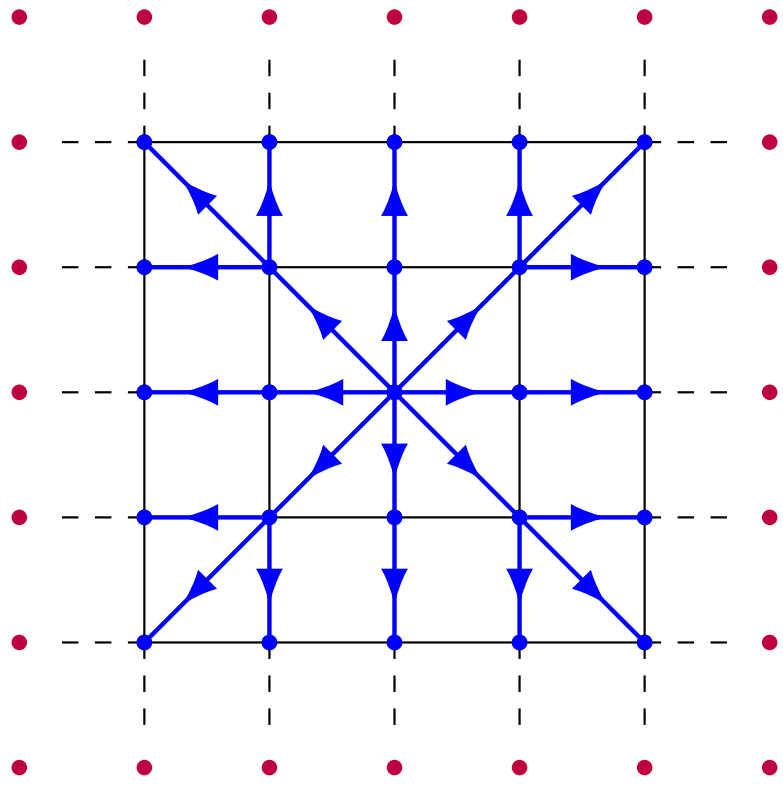
The boundaries ∂X and ∂L are homeomorphic.



Equivalence of boundaries

Theorem

The boundaries ∂X and ∂L are homeomorphic.



Boone-Higman Conjecture (now an OpenAI Theorem?)

Conjecture (1973)

A finitely generated group has solvable word problem if and only if it embeds in a finitely presented simple group.

Boone-Higman Conjecture (now an OpenAI Theorem?)

Conjecture (1973)

A finitely generated group has solvable word problem if and only if it embeds in a finitely presented simple group.

Theorem (Belk, Bleak, M, Zaremsky, 2026)

Every hyperbolic group embeds in a finitely presented simple group.

Boone-Higman Conjecture (now an OpenAI Theorem?)

Conjecture (1973)

A finitely generated group has solvable word problem if and only if it embeds in a finitely presented simple group.

Theorem (Belk, Bleak, M, Zaremsky, 2026)

Every hyperbolic group embeds in a finitely presented simple group.

OpenAI says that the BH conjecture is true... So what now?

Boone-Higman Conjecture (now an OpenAI Theorem?)

Conjecture (1973)

A finitely generated group has solvable word problem if and only if it embeds in a finitely presented simple group.

Theorem (Belk, Bleak, M, Zaremsky, 2026)

Every hyperbolic group embeds in a finitely presented simple group.

OpenAI says that the BH conjecture is true... So what now?

Natural Boone-Higman embeddings: given a group with solvable WP, find natural f.p. simple groups containing it (just like Higman asked for natural embeddings of recursively enumerable groups, e.g. $\mathbb{Q}$).

Boone-Higman Conjecture (now an OpenAI Theorem?)

Conjecture (1973)

A finitely generated group has solvable word problem if and only if it embeds in a finitely presented simple group.

Theorem (Belk, Bleak, M, Zaremsky, 2026)

Every hyperbolic group embeds in a finitely presented simple group.

OpenAI says that the BH conjecture is true... So what now?

Natural Boone-Higman embeddings: given a group with solvable WP, find natural f.p. simple groups containing it (just like Higman asked for natural embeddings of recursively enumerable groups, e.g. $\mathbb{Q}$).

Geometric Boone-Higman embeddings: The hyperbolic groups proof of Boone-Higman is geometric, while the OpenAI proof is algebraic and it would be pleasant to find a geometric proof.

Open questions about continuous automaticity

Question 1. Are hyperbolic groups continuous automatic? And bi-automatic groups?

Open questions about continuous automaticity

Question 1. Are hyperbolic groups continuous automatic? And bi-automatic groups?

Question 2. What about closure properties? Automatic groups are closed under taking amalgams/HNN extensions over finite groups or graph products.

Open questions about continuous automaticity

Question 1. Are hyperbolic groups continuous automatic? And bi-automatic groups?

Question 2. What about closure properties? Automatic groups are closed under taking amalgams/HNN extensions over finite groups or graph products.

Question 3. Does every automatic group admit a continuous automatic structure?

Thank you!