Groups with dual classes of subgroups

(a joint work with M. Trombetti and A. Zitouni)

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Introduction

From subgroup conditions to group structure

Problem

Let $\mathcal{P}$ be a subgroup property. Classify all the groups G such that every (proper) subgroup H of G has the property $\mathcal{P}$.

Dedekind's theorem

Theorem (Dedekind, 1897)

A finite group G has all subgroups normal if and only if one of the following hold:

- *G is abelian;*
- *$G \cong Q_8 \times E \times A$, where E is an elementary abelian 2-group, and A is an abelian group of odd order.*



R. Dedekind, *Ueber Gruppen, deren sämmtliche Theiler Normaltheiler sind*, Math. Ann. **48** (1897), 548–561.

From normality to weaker embedding properties

Normality is extremely restrictive. The next step is to weaken normality, but still keep some relation between subgroups.

$$\text{Normality} \quad \rightsquigarrow \quad \text{Pronormality}.$$

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The concept of pronormality was introduced by P. Hall in his lectures in Cambridge.

First results on pronormal subgroups appeared in



J.S. Rose, *Finite soluble groups with pronormal system normalizers*, Proc. London Math. Soc. **17** (1967), 447–469

Examples

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  $H^x = H$ for all $x \in G$.
  Therefore, H^x and H are conjugate (for example by 1) in $\langle H, H^x \rangle$.

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  - $M^x = M$ (trivial case);
  - $\langle M, M^x \rangle = G \ni x$.

Examples

- Normal subgroups are pronormal ($H \triangleleft G$);
- Maximal subgroups are pronormal ($M \lessdot G$)
- Sylow p-subgroups in finite groups are pronormal.

Basic results

$$H \text{ pronormal in } G \quad + \quad H \text{ subnormal in } G \quad \Longrightarrow \quad H \trianglelefteq G$$

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Example

$G = \mathrm{D}_8 = \langle x, a : x^2 = a^4 = 1, a^x = a^{-1} \rangle$;

The subgroup $H = \langle x \rangle$ is NOT pronormal in G.

Groups in which every subgroup is pronormal

Theorem (Gaschütz, 1957 –Peng, 1969)

Let G be a finite group. Then every subgroup of G is pronormal in G if and only if G has an abelian normal subgroup A satisfying:

1. *G/A is a Dedekind group;*
2. *$\pi(A) \cap \pi(G/A) = \varnothing$ and $2 \notin \pi(A)$;*
3. *every element of G induces a power automorphism of A, and $A = [A, G]$;*
4. *$G = A \rtimes Q$, where Q is any Sylow $\pi(G/A)$-subgroup of G.*

Locally graded groups

Definition

A group G is said to be *locally graded* if every nontrivial finitely generated subgroup of G has a nontrivial finite quotient.

Introduced by Černikov in 1970 to provide a broad class containing locally finite, locally soluble and residually finite groups, while excluding finitely generated infinite simple groups.

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Closure properties:

The class of locally graded groups is closed under **subgroups** and **extensions**.

Not preserved under arbitrary **quotients**.

Theorem (Robinson–Russo–Vincenzi, 2007)

Let G be a locally graded group. Then every subgroup of G is pronormal in G if and only if one of the following conditions holds:

1. *G is abelian;*
2. *G is periodic and has an abelian normal subgroup A satisfying:*
   - 2.1 *G/A is a Dedekind group;*
   - 2.2 *$\pi(A) \cap \pi(G/A) = \varnothing$ and $2 \notin \pi(A)$;*
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This theorem extends the classification of locally finite groups obtained by **Kuzennyi and Subbotin** in 1987.

From one subgroup property to two

Problem

Classify all groups G such that

$$\forall H \leq G \ \rightsquigarrow \ H \text{ is pronormal in } G \text{ or } H \leq N_{pro}(G).$$

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- $Z(G) \leq N_{pro}(G)$;
- If $G = N_{pro}(G)$, then every subgroup of G is pronormal;

Examples

Let's look at some examples of finite groups every subgroup is either pronormal or contained in the pronorm.

Examples:

- S_3

Non-examples:

- D_8

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$G = A_5$

Maximal subgroups, up to conjugacy:

$$\boxed{A_4, \qquad D_{10}, \qquad S_3}$$

$N_{pro}(A_5) = \{1\} \cup \{g \mid o(g) = 2\} \cup \{g \mid o(g) = 5\}$

$|N_{pro}(A_5)| = 40.$

p-groups

Classification of p-groups

Theorem

Let G be a non-abelian locally graded p-group. Then every subgroup of G is pronormal or contained in the pro-norm of G if and only if one of the following conditions holds:

1. $G = Q_8 \times E$, *where E is an elementary abelian 2-group;*
2. $G = \langle x \rangle \ltimes (\langle a \rangle \times B)$, *where $B \leq Z(G)$,*

$$a^x = a^{1+p^s}, \qquad G' = \langle a^{p^s} \rangle,$$

with $s \geq 1$ (and $s \geq 2$ if $p = 2$), and

$$o(a) \geq \frac{|G'|^2}{p} \max\{o(x), \exp(B)\}.$$

Soluble groups

Soluble groups

Theorem

Let G be a soluble group such that every subgroup is pronormal or contained in $N_{\mathrm{pro}}(G)$. Then G has derived length at most 3.

Example: $G = Q_8 \ltimes (C_3 \times C_3)$ has derived length 3.

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Every such finite soluble group has a Sylow tower.

Insoluble groups

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Which finite simple groups have Sylow 2-subgroups of nilpotency class at most 2?

Simple groups with Sylow 2-subgroups of class 2

$\mathrm{PSL}(2,q),\ q \equiv_{16} 7,9$

A_7

$\mathrm{Sz}(2^n),\ n \geq 3$ odd

$\mathrm{PSL}(3,2^n),\ n \geq 2$

$\mathrm{PSp}(4,2^n),\ n \geq 2$

$\mathrm{U}(3,2^n),\ n \geq 2$

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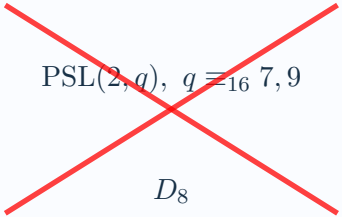
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Simple groups with abelian Sylow 2-subgroups

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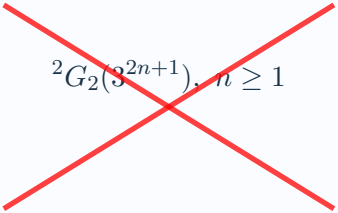
${}^2G_2(3^{2n+1}),\ n \geq 1$

J_1

Simple groups with abelian Sylow 2-subgroups

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$\mathrm{PSL}(2, 2^n),\ n \geq 2$



${}^2G_2(3^{2n+1}),\ n \geq 1$

J_1

Theorem

Let G be a non-abelian finite simple group. Then every subgroups of G is pronormal or contained in the pronorm if and only if G is isomorphic to one of the following types of groups:

(1) $\mathrm{PSL}(2, 5^m)$, *where either* $m = 1$ *or* m *is an odd prime*;

(2) $\mathrm{PSL}(2, q)$, *where* q *is a prime with* $q > 5$ *and* $q \equiv_8 3, 5$;

(3) $\mathrm{PSL}(2, 2^m)$, *where* m *is a prime*;

(4) *the first Janko group* J_1.

Infinite Simple Groups

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Insoluble groups

Theorem

Let G be a locally graded insoluble group. Then the subgroups of G are pronormal or contained in the pro-norm of G if and only if one of the following possibilities holds:

(1) $G = H \times K$, *where*:

- H *is periodic and all its subgroups of* H *are pronormal,*
- $\pi(H) \cap \pi(K) \subseteq \{2\}$,
- *if* $Z(K) \neq \{1\}$, *then the Sylow* 2*-subgroups of* H *are elementary abelian, and*
- K *is isomorphic to one of the following groups*: J_1, $\mathrm{PSL}(2, 2^p)$, *with* p *a prime, or* $\mathrm{PSL}(2, q)$, $\mathrm{SL}(2, q)$, *where*:
  - $q = 5^m$ *with* $m = 1$ *or* m *an odd prime, or*
  - $q > 5$ *and* $q \equiv_8 3, 5$, *or*
  - $q = 2^m$, *where* m *is a prime.*

Theorem (continued)

(2) $G = C \ltimes (H \times K)$, where:

- $K = \mathrm{PSL}(2, 2^p)$ for a prime $p > 3$,
- $[C, K] \neq \{1\}$ and C is cyclic and has order p^n for some integer $n \geq 1$,
- $\pi(H) \cap \pi(K) \subseteq \{2\}$,
- $H, K \trianglelefteq G$, and
- H is periodic, and subgroups of $C \ltimes H$ are pronormal.


Thank
you