The prime graph of finite solvable groups that are rational or cut

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Graphs defined on groups

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Let G be a finite group. Elements $x \neq y$ of G are joined in the

- *power graph* if one is a power of the other;
- *enhanced power graph* if $\langle x, y \rangle$ is cyclic;
- *commuting graph* if $xy = yx$;
- *non-generating graph* if $\langle x, y \rangle \neq G$.

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If G is non-abelian, each edge set is contained in the next one.

[Cameron, 2022]

The Gruenberg-Kegel graph

Definition

The Gruenberg-Kegel graph (or prime graph) $\Gamma_{\mathrm{GK}}(G)$ has

- vertices: the primes dividing $|G|$,
- an edge p–q, $p \neq q$, if G has an element of order pq.

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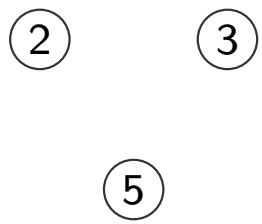
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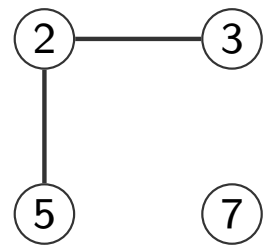
S_4

no element of order 6



A_5

orders $1, 2, 3, 5$



S_7

orders $6, 10$; not $14, 15, 21, 35$

Why the Gruenberg-Kegel graph?

- The augmentation ideal of $\mathbb{Z}G$ is indecomposable as a $\mathbb{Z}G$-module iff $\Gamma_{\mathrm{GK}}(G)$ is connected. [Gruenberg-Roggenkamp, 1975; Williams, 1981]

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- $\Gamma_{\mathrm{GK}}(G)$ and $\Gamma_{\mathrm{GK}}(V(\mathbb{Z}G))$ have the same vertices, where $V(\mathbb{Z}G)$ is the group of units of augmentation 1. [Higman, 1940 (thesis)]

Solvable groups

Theorem (Gruenberg-Kegel; see Williams, 1981)

If G is solvable, then $\Gamma_{\mathrm{GK}}(G)$ has at most two connected components, and it is disconnected iff G is a Frobenius or a 2-Frobenius group.

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Lemma (Higman, 1957; Lucido, 1999)

If G is solvable, then among any three vertices of $\Gamma_{\mathrm{GK}}(G)$ two are adjacent. Each connected component has diameter at most 3.

Realizable graphs

Theorem (Gruber-Keller-Lewis-Naughton-Strasser, 2015)

An unlabelled graph is $\Gamma_{\mathrm{GK}}(G)$ for some solvable G iff its complement is triangle-free and 3-colourable.

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Studied for:

- T-solvable groups, T a small simple group [Keller et al., 2023–2025]
- complete bipartite graphs [Maslova-Pagon, 2016]
- $|\pi(G)| \leq 5$ [Gavrilyuk-Khramtsov-Kondrat'ev-Maslova, 2014]

Trivial central units

Units of integral group rings

Theorem (Higman, 1940)

All units of $\mathbb{Z}G$ are trivial, i.e. $\mathcal{U}(\mathbb{Z}G) = \pm G$, iff G is abelian of exponent dividing 4 or 6, or $G \cong Q_8 \times C_2^n$.

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Question

When is $\rho(G) = 0$? When is $\rho(G) \leq 1$?

The rank of the central units

Theorem (Ferraz, 2004)

$$\begin{aligned}\rho(G) &= \#\{\mathbb{R}\text{-classes of } G\} - \#\{\mathbb{Q}\text{-classes of } G\} \\ &= \#\{\text{simple comp. of } \mathbb{R}G\} - \#\{\text{simple comp. of } \mathbb{Q}G\}.\end{aligned}$$

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Groups with $\rho(G) \leq 1$: extended cut groups.

[Bovdi-Breuer-Maróti, 2020; García-Blázquez-Kaur-Maheshwary, 2025]

Cut groups

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The following are equivalent:

1. G is cut.
2. $\mathbb{Q}(\chi)$ is $\mathbb{Q}$ or imaginary quadratic for every $\chi \in \mathsf{Irr}(G)$.
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Examples: C_3, C_4 are cut; C_{12} is not.

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With $B_G(g) = N_G(\langle g \rangle)/C_G(g) \leq \mathrm{Aut}(\langle g \rangle)$:

- rational $\iff$ $B_G(g) = \mathrm{Aut}(\langle g \rangle)$ for all g,
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Rational $\Rightarrow$ cut, but among groups of order ≤ 512: 86.62% are cut, 0.57% are rational. [Bächle-Caicedo-Jespers-Maheshwary, 2021]

Small character fields

Rows and columns of the character table:

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) : g \in G), \qquad \mathbb{Q}(g) = \mathbb{Q}(\chi(g) : \chi \in \mathsf{Irr}(G)).$$

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$$\text{rational} \subsetneq \text{cut} \subsetneq \text{uniformly semi-rational}$$
$$\subsetneq \text{semi-rational} \cap \text{character-quadratic}$$

Primes

Theorem

Let G be solvable. Then $\pi(G)$ is contained in

- $\{2, 3, 5\}$ if G is rational [Gow, 1976]
- $\{2, 3, 5, 7\}$ if G is cut [Bächle, 2018]
- $\{3, 7\}$ if G is cut of odd order [Maheshwary, 2018]
- $\{2, 3, 5, 7, 13\}$ if G is character-quadratic [Tent, 2012]
- $\{2, 3, 5, 7, 13, 17\}$ if G is semi-rational [Chillag-Dolfi, 2010]

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- $\{2, 3, 5, 7, 13, 17\}$ if G is semi-rational [Chillag-Dolfi, 2010]

- All these primes occur, except that 17 is open.
- Nilpotent cut groups are $\{2, 3\}$-groups.

The structure of rational groups

Theorem (Hegedüs, 2005)

If G is solvable and rational, then the Sylow 5-subgroup of G is normal and elementary abelian.

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Non-abelian composition factors:

- rational groups: alternating groups and a finite list [Feit-Seitz, 1989; Thompson, 2008]
- cut groups: an explicit list [Trefethen, 2019]

The Prime Graph Question

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Question (Kimmerle, 2006)

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Theorem (Bächle-Kiefer-Maheshwary-del Río, 2023)

$\Gamma_{\mathrm{GK}}(V(\mathbb{Z}G)) = \Gamma_{\mathrm{GK}}(G)$ for cut groups with no epimorphism onto the Monster. In particular, yes for rational groups.

Prime graphs of solvable rational and cut groups

A classification problem

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Lemma (Bächle-Kiefer-Maheshwary-del Río, 2023)

If G is solvable and cut, then $\pi(G)$ is one of

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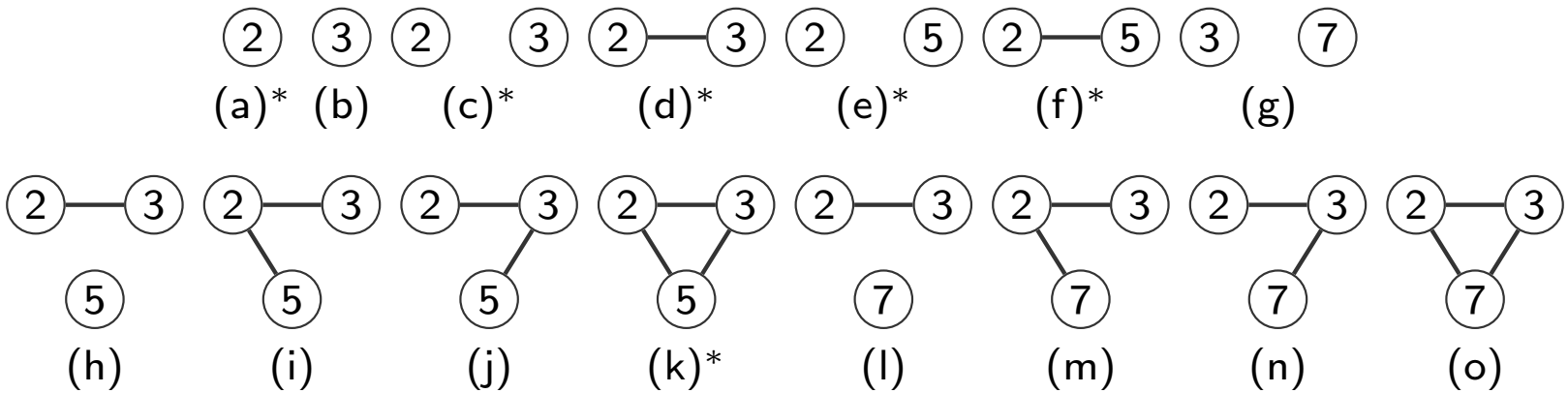
$$\{2\},\ \{3\},\ \{2,3\},\ \{2,5\},\ \{3,7\},\ \{2,3,5\},\ \{2,3,7\},\ \{2,3,5,7\}.$$

Indeed, if $|g| = 5$, then $B_G(g) \cong C_4$; if $|g| = 7$, then 3 divides $|B_G(g)|$.

At most three vertices

Theorem (Bächle-Kiefer-Maheshwary-del Río, 2023)

A graph with at most 3 vertices is $\Gamma_{\mathrm{GK}}(G)$ for a non-trivial solvable cut group G iff it is one of:

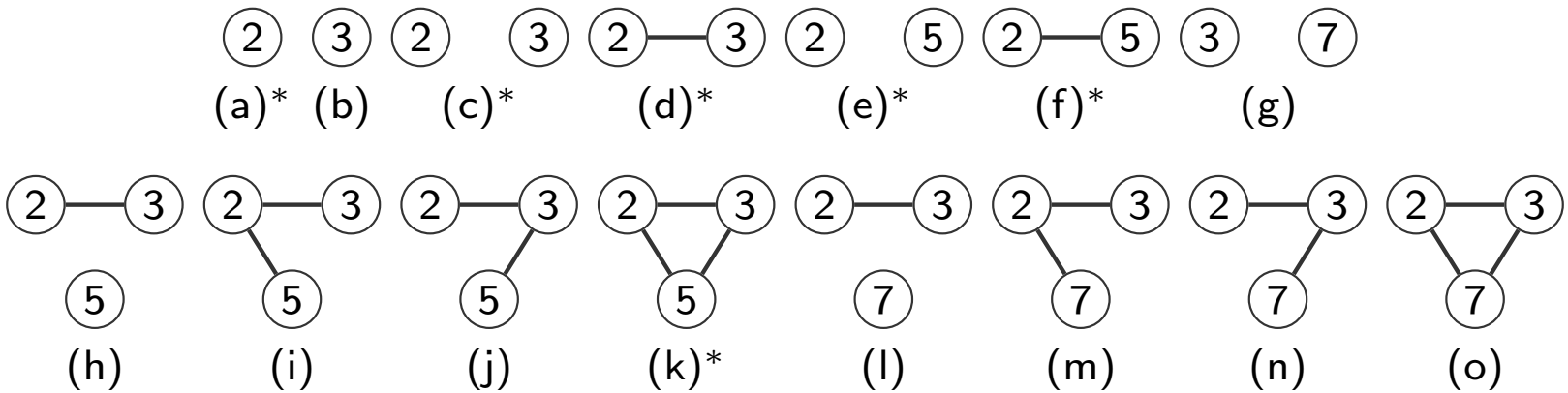


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Corollary

A graph with 3 vertices occurs iff 2–3 is an edge and 5 or 7 is a vertex.

Realizing groups

(a)	C_2	(j)	$M \times C_3$
(b)	C_3	(k)	$M \times S_3$
(c)	S_3	(l)	$C_7 \rtimes C_6$
(d)	$S_3 \times C_2$	(m)	$(C_7 \rtimes C_6) \times C_2$
(e)	$M = C_5^2 \rtimes Q_8$	(n)	$(C_7 \rtimes C_6) \times C_3$
(f)	$M \times C_2$	(o)	$(C_7 \rtimes C_3) \times S_3$
(g)	$C_7 \rtimes C_3$	(p)	$M \times (C_7 \rtimes C_3)$
(h)	$C_5^2 \rtimes (C_3 \rtimes C_4)$	(q)	$M \times (C_7 \rtimes C_3) \times C_2$
(i)	$(C_5^2 \rtimes (C_3 \rtimes C_4)) \times C_2$	(r)	$M \times (C_7 \rtimes C_6) \times C_3$

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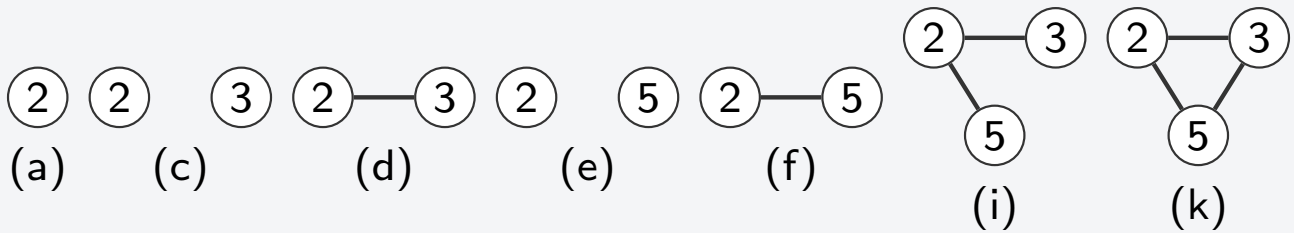
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Rational: (a), (c), (d), (e), (f), (k).
A direct product of a rational group and a cut group is cut.

Solvable rational groups

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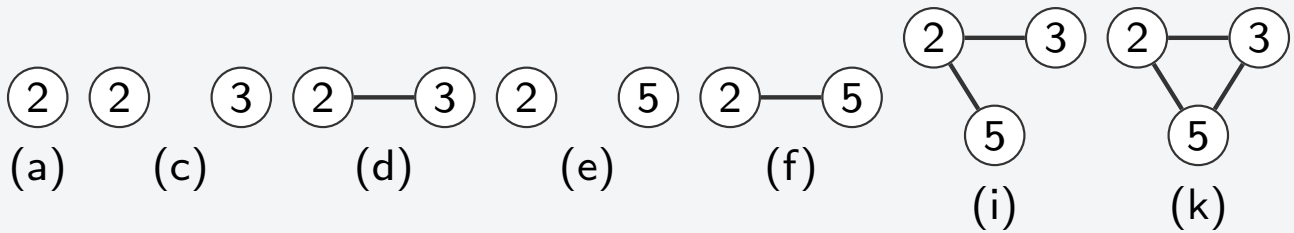


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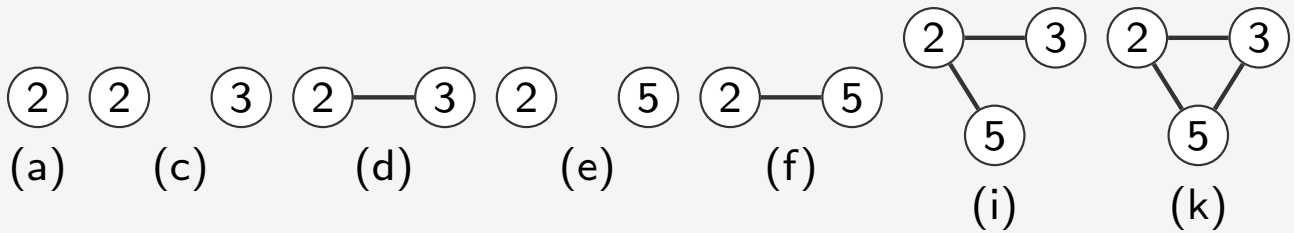
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Question (Bächle-Kiefer-Maheshwary-del Río, 2023)

Is 3–2–5 the prime graph of a finite solvable rational group?

The GK graph of solvable rational groups

Theorem (Debón-GL-del Río, 2024)

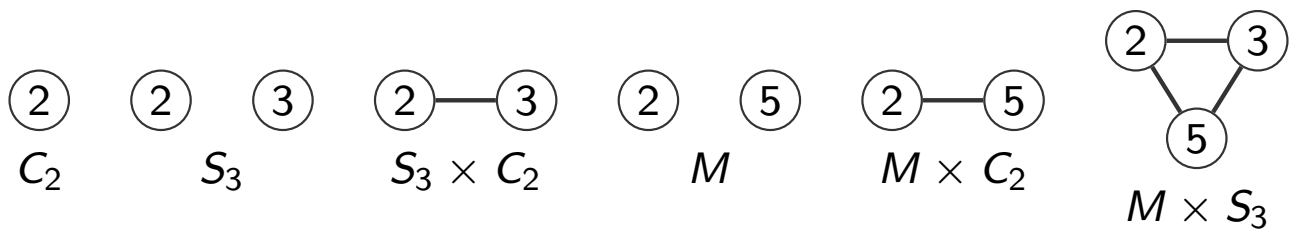
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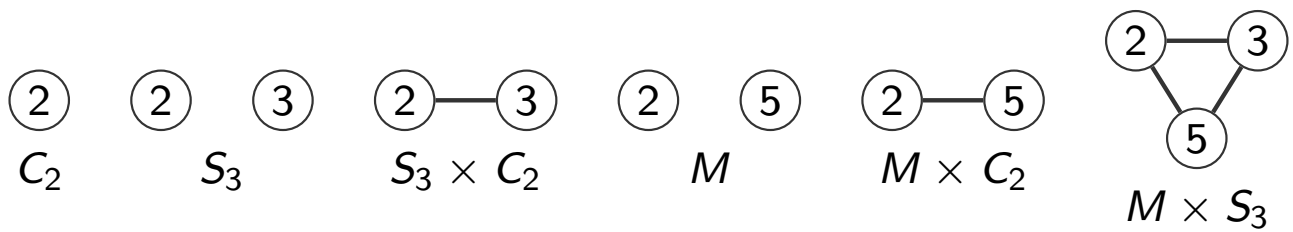


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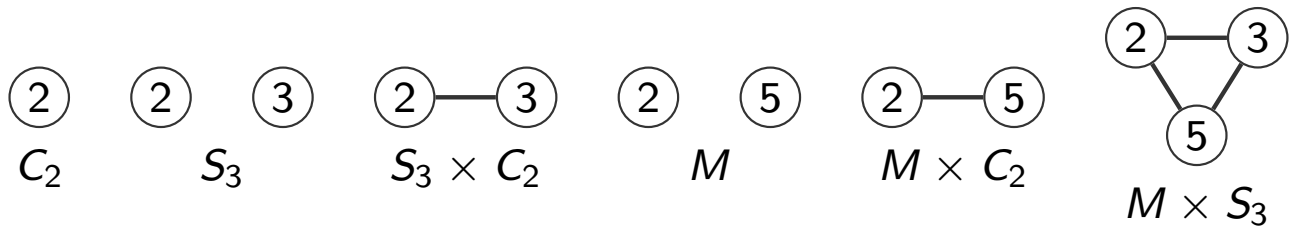
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If G is solvable, rational and $15 \mid |G|$, then G has elements of orders 6, 10 and 15.

But 3–2–5 is the prime graph of the cut group $(C_5^2 \rtimes (C_3 \rtimes C_4)) \times C_2$.

Idea of the proof

Let G be a minimal solvable rational group with $\Gamma_{\mathrm{GK}}(G) = 3$–$2$–$5$.

- Hegedüs: $G = V \rtimes S$, V the Sylow 5-subgroup, elementary abelian and minimal normal; S acts faithfully.

Idea of the proof

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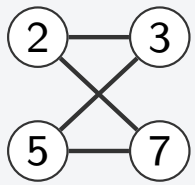
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- Each case is excluded: orbit arguments against the eigenvector property; for $\mathrm{SL}(2,3)$, we construct an *ad hoc* vector of V that cannot be "rationalized".

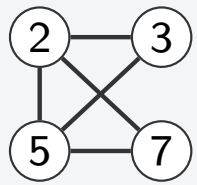
Four vertices

Theorem (Bächle-Kiefer-Maheshwary-del Río, 2023)

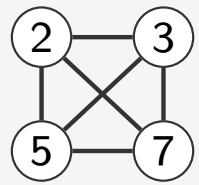
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(p)



(q)

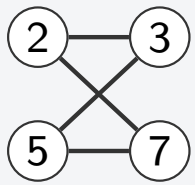


(r)

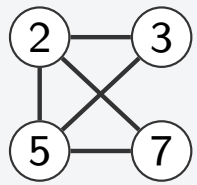
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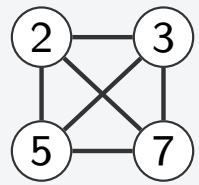
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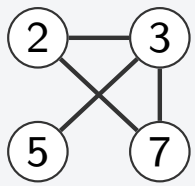


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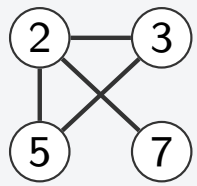


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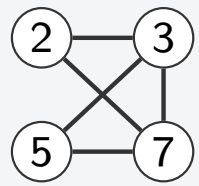
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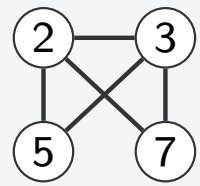
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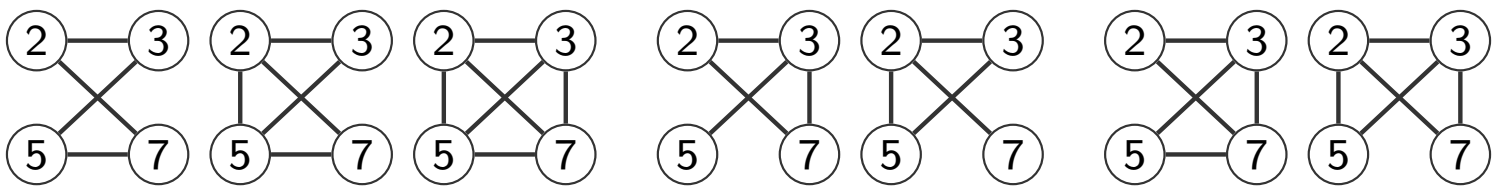
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State of the art with 4 vertices:



(p) (q) (r) (s) no (t) no (u) ? (v) ?

Ideas for (s) and (t)

Minimal counterexample: $G = V \rtimes S$, V an elementary abelian Sylow p-subgroup ($p = 7$ for (s), $p = 5$ for (t)), S faithful and irreducible on V.

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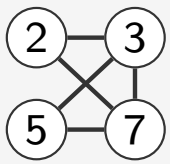
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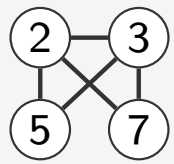
Theorem (Debón, 2026)

If a rational 2-group Q has a faithful $\mathbb{F}_p Q$-module, $p \geq 5$, with the eigenvector property, then $p = 5$ and $Q \cong Q_8 \wr K$ for a 2-group $K \leq S_n$.

Open(?) problem

Question

Are (u) 

or (v) 

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Three questions on cut groups

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These three questions are now in the Kourovka Notebook (20.4).

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A counterexample must involve arbitrarily large alternating groups.

Question 2: Sylow 3-subgroups

For the prime 2 the answer is negative:

- Sylow 2-subgroups of rational groups need not be rational.
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In general, P/P' is elementary abelian. [Navarro-Tiep, 2019]

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- N is not rational: by Hegedűs for $p = 5$, by Gow for $p = 7$.

Thanks for your attention.

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