Commuting probability of conjugate subgroups

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Joint work with R. M. Guralnick, M. Morigi and P. Shumyatsky [C],[A],[P]

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Commuting probability of subgroups

Let H, K be subgroups of a finite group G. Set

$$\Pr(H,K) = \frac{|\{(h,k) \in H \times K \mid [h,k] = 1\}|}{|H|\,|K|}.$$

Thus $\Pr(H,K) = 1$ if and only if $[H,K] = 1$.

P. M. Neumann, 1989

If $\Pr(G,G) \geq \epsilon > 0$, then G has a normal subgroup R with

$$|G:R| \quad \text{and} \quad |R'| \qquad \epsilon\text{-bounded}.$$

In other words, G is bounded-by-abelian-by-bounded.

Throughout, ϵ-bounded means bounded by a function of ϵ only.

- Gustafson, 1973

$$\Pr(G,G) > \frac{5}{8} \quad \Rightarrow \quad G \text{ is abelian}$$

The bound is attained in D_8 and Q_8.

- P. Lescot, 1995

$$\Pr(G,G) > \frac{1}{2} \quad \Rightarrow \quad G \text{ is nilpotent}$$

- Dixon, 1973, P. Lescot, 1988-89

$$\Pr(G,G) > \frac{1}{12} \quad \Rightarrow \quad G \text{ is soluble}$$

The bound is attained in $\mathrm{Alt}(5)$.

- Guralnick and Robinson, 2006

$$\Pr(G,G) > \frac{3}{40} \quad \Rightarrow \quad G \text{ is solvable or } G = \mathrm{Alt}(5) \times A, \text{ with } A \text{ abelian.}$$

Two comparisons

Conjugate Sylow subgroups

Can probabilities involving Sylow subgroups give comparable information?

Fix $P \in \mathrm{Syl}_p(G)$ and assume

$$\Pr(P, P^x) \geq \epsilon \qquad \text{for every } x \in G.$$

How large can P be outside $O_p(G)$?

Sylow subgroups of different primes

For each $p \neq q$, find $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with

$$\Pr(P, Q) \geq \epsilon.$$

How much of G is accounted for by its Fitting series?

We will return to both comparisons for coprime automorphisms and for profinite groups.

Conjugate Sylow subgroups

Let G be finite and $P \in \mathrm{Syl}_p(G)$. The subgroup $O_p(G)$ is the largest normal p-subgroup of G.

D.–Guralnick–Morigi–Shumyatsky [C]

If

$$\Pr(P, P^x) \geq \epsilon > 0 \qquad \text{for every } x \in G,$$

then $|P : O_p(G)|$ is ϵ-bounded. Moreover, $O_p(G)$ is bounded-by-abelian-by-bounded.

Why expect the normal p-subgroup to matter?

If $B \lhd G$ is an abelian p-subgroup, then $B \leq P \cap P^x$, so

$$\Pr(P, P^x) \geq |P : B|^{-2}.$$

Why the Sylow hypothesis matters

Let $G = \mathrm{SL}(n, p)$ acting on the natural module V and let P be the elementary abelian group of order p^{n-1} acting trivially on a hyperplane U and on V/U.
For every $x \in G$, the groups P and P^x contain subgroups of index p which centralize each other. Hence

$$\Pr(P, P^x) \geq \frac{1}{p^2} \qquad \text{for every } x \in G.$$

However,

$$O_p(G) = 1, \qquad |P| = p^{n-1} \longrightarrow \infty.$$

Thus the conclusion is false for arbitrary p-subgroups.

The role of characteristic p

Theorem [C]

Let P be any p-subgroup of a finite group G, with

$$\Pr(P, P^x) \geq \epsilon > 0 \qquad (x \in G).$$

Assume that the composition factors of Lie type in characteristic p have Lie rank at most r. Then

$$|P : P \cap O_p(G)| \text{ is } (r, \epsilon)\text{-bounded.}$$

In particular [C]

If G is p-soluble, then $|P : P \cap O_p(G)|$ is ϵ-bounded.

A finite group is p-soluble if its composition factors are p-groups or p'-groups.

Tools in the finite proof

Assume $\Pr(P, P^x) \geq \epsilon > 0$ for all $x \in G$.

Centralizers control powers [C]

$$\Pr(H, K) = \frac{1}{|H|} \sum_{h \in H} \frac{1}{|K : C_K(h)|}$$

Large subgroups have bounded centralizer indices. Applying this to all conjugates of P yields

$$P^e \leq O_p(G) \qquad (e \ \epsilon\text{-bounded}).$$

Decay across independent factors

For pairs in r direct factors, if $[H_i, K_i] \neq 1$, then

$$\Pr\left(\prod_{i=1}^{r} H_i, \prod_{i=1}^{r} K_i\right) = \prod_{i=1}^{r} \Pr(H_i, K_i) \leq (3/4)^r.$$

Different primes: the Fitting series

D.–Lucchini–Morigi–Shumyatsky, 2025

If for every $p \neq q$ in $\pi(G)$ there are $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with $\Pr(P, Q) \geq \epsilon > 0$, then

$$|G : F_2(G)| \text{ is } \epsilon\text{-bounded.}$$

Here $F_2(G)/F(G) = F(G/F(G))$.

The index of $F(G)$ need not be bounded

Let $G = \prod_{i=1}^{s} D_{2p_i}$, where the p_i are distinct odd primes and $|D_{2p_i}| = 2p_i$.

$$\Pr(S_2, S_{p_i}) = \frac{p_i + 1}{2p_i} > \frac{1}{2}, \qquad |G : F(G)| = 2^s.$$

Sylow subgroups of distinct odd orders commute.

Coprime action

Let a finite group A act on the finite group G, with $(|A|, |G|) = 1$.

$$[G, A] = \langle g^{-1} g^a \mid g \in G,\ a \in A \rangle.$$

The action and its fixed points

$$G = [G, A] C_G(A), \qquad [G, A, A] = [G, A].$$

There are A-invariant Sylow subgroups for every prime.

For an A-invariant Sylow subgroup P, we compare the subgroup $[P, A]$ with its conjugates, or with $[Q, A]$ for another prime.

The structural conclusions will concern $[G, A]$.

The finite results extend to $[P, A]$

Let G be finite, A act coprimely, and $\epsilon > 0$. All Sylow subgroups in this slide are A-invariant.

Conjugates – D.–Guralnick–Morigi–Shumyatsky [A]

For $P \in \mathrm{Syl}_p(G)$, if

$$\Pr([P, A], [P, A]^x) \geq \epsilon \qquad \forall x \in G,$$

then $|[P, A] : [P, A] \cap O_p(G)|$ is ϵ-bounded.

If this condition holds for every prime p, then $[G, A]$ is bounded-by-abelian-by-bounded.

Different primes [A]

If for every $p \neq q$ there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with

$$\Pr([P, A], [Q, A]) \geq \epsilon,$$

then $|[G, A] : F_2([G, A])|$ is ϵ-bounded.

Positivity in a profinite group

For closed subgroups H, K of a profinite group G, define

$$\Pr(H,K) = (\mu_H \times \mu_K)\{(h,k) \in H \times K \mid [h,k] = 1\},$$

using normalized Haar measures. Then

$$\Pr(H,K) = \inf_{N \lhd_o G} \Pr\left(\frac{HN}{N}, \frac{KN}{N}\right).$$

The profinite question

Assume only

$$\Pr(P, P^x) > 0 \qquad \forall x \in G.$$

No common positive lower bound is assumed.

What structural conclusions follow from this pointwise positivity?

Conjugate Sylow subgroups: profinite case

Theorem [C]

Let $P \in \mathrm{Syl}_p(G)$, with G profinite. If

$$\Pr(P, P^x) > 0 \qquad \forall x \in G,$$

then $O_{p,p'}(G)$ is open, and $O_p(G)$ is virtually abelian.

Here

$$O_{p,p'}(G)/O_p(G) = O_{p'}(G/O_p(G)).$$

Thus an open normal subgroup is pro-p-by-pro-p'. For the proof,

$x = 1$ gives $\Pr(P, P) > 0$, hence P is virtually abelian and therefore soluble.

This controls the normal series; probability estimates control its semisimple sections.

One odd prime and pro-p-solubility

D.–Guralnick–Morigi–Shumyatsky[P]

Let G be profinite, $L \in \mathrm{Syl}_2(G)$ and $P \in \mathrm{Syl}_p(G)$, with p odd.

$$\Pr(L, P) > 0 \quad \Longrightarrow \quad G \text{ is virtually pro-}p\text{-soluble.}$$

A pro-p-soluble group is an inverse limit of finite p-soluble groups.

The primes 3 and 5 (D.–Morigi–Shumyatsky)

For $L \in \mathrm{Syl}_2(G)$, $P \in \mathrm{Syl}_3(G)$ and $Q \in \mathrm{Syl}_5(G)$,

$$\Pr(L, P) > 0, \quad \Pr(L, Q) > 0 \quad \Longrightarrow \quad G \text{ is virtually prosoluble.}$$

A finite group which is both 3-soluble and 5-soluble is soluble.

Fixed points of a coprime action

Let a finite group A act coprimely on a profinite group G.

The general form [P]

Let

$$T \in \mathrm{Syl}_2(C_G(A)), \qquad P \in \mathrm{Syl}_p(G), \qquad p \text{ odd}.$$

If $\Pr(T, P) > 0$, then G is virtually pro-p-soluble.

The subgroup T lies in the fixed-point group; P is a Sylow subgroup of the whole group G.

No A-invariance of P is required.

Taking $A = 1$ recovers the previous theorem.

Two tests inside $C_G(A)$

Let a finite group A act coprimely on a profinite group G.

Theorem [P]

Suppose $C_G(A)$ has Sylow subgroups

$$T \in \mathrm{Syl}_2(C_G(A)), \quad U \in \mathrm{Syl}_3(C_G(A)), \quad V \in \mathrm{Syl}_5(C_G(A))$$

such that

$$\Pr(T, U) > 0, \qquad \Pr(T, V) > 0.$$

Then G is virtually prosoluble.

Both probabilities are measured inside $C_G(A)$, but the conclusion concerns G.

For a single odd prime, a test using only fixed points is insufficient (example).

The key estimate on simple factors

The simple-group input [P]

If S is finite nonabelian simple and α is coprime, no Sylow p-subgroup of S, for odd $p \mid |S|$, centralizes a Sylow 2-subgroup of $C_S(\alpha)$.

For $G = S_1 \times \cdots \times S_t$, with $p \mid |S_i|$ for every i, and a coprime action of a finite group A, take $T \in \mathrm{Syl}_2(C_G(A))$ and any $P \in \mathrm{Syl}_p(G)$. Then

$$\Pr(T, P) \leq (3/4)^{t/|A|}.$$

Each A-orbit has at most $|A|$ factors and contributes at most $3/4$.

A lower bound $\eta > 0$ bounds t. In the proof this controls non-p-soluble length in finite images and makes the semisimple sections finite.

Conjugates of $[P, A]$: profinite case

Let G be profinite, A a finite group of coprime automorphisms, and $P \in \mathrm{Syl}_p(G)$ an A-invariant Sylow subgroup. Assume

$$\Pr([P, A], [P, A]^x) > 0 \qquad \forall x \in G.$$

The local conclusion [P]

$$[P, A]O_p(G)/O_p(G) \text{ is finite.}$$

The global conclusion [P]

The group $[G, A]$ has an open normal subgroup N with

$$\lambda_p(N) \leq 1.$$

Here λ_p is the minimum number of layers consisting of Cartesian products of nonabelian finite simple groups of order divisible by p, in a normal series whose other factors are pro-p-soluble.

Why one simple layer really is necessary

Example [P]

Take

$$S = \mathrm{PSL}(2, 3^5), \qquad G = \prod_{i=1}^{\infty} S,$$

and let α act on every factor as a field automorphism of order 5. It is coprime, and $C_S(\alpha) \cong \mathrm{PSL}(2, 3)$. An α-invariant Sylow 2-subgroup P of G is fixed pointwise. Thus

$$[P, \alpha] = 1, \qquad \Pr([P, \alpha], [P, \alpha]^x) = 1 \quad \forall x \in G.$$

Yet $[G, \alpha] = G$, and every open subgroup contains infinitely many simple factors. Hence G is not virtually prosoluble.

Here $\lambda_2(G) = 1$.

Further results: different primes

D.–Guralnick–Morigi–Shumyatsky [P]

Let G be profinite and let a finite group A act coprimely on G. Suppose that $L \in \mathrm{Syl}_2(G)$ and $P \in \mathrm{Syl}_p(G)$ are A-invariant, where p is odd, and that

$$\Pr([L, A], [P, A]) > 0.$$

Then $[G, A]$ has an open normal subgroup N such that

$$\lambda_p(N) \leq 1.$$

The example with $\prod \mathrm{PSL}(2, 3^5)$ also shows that $[G, A]$ need not be virtually pro-p-soluble.

What the conjugate probabilities control

Hypothesis	Conclusion
Finite G, $P \in \mathrm{Syl}_p(G)$, $\Pr(P, P^x) \geq \epsilon > 0$ for all x	$\lvert P : O_p(G)\rvert$ is ϵ-bounded.
Profinite G, $P \in \mathrm{Syl}_p(G)$, $\Pr(P, P^x) > 0$ for all x	$O_{p,p'}(G)$ is open.
Profinite G, finite coprime A, $P \in \mathrm{Syl}_p(G)$ is A-invariant, $\Pr([P, A], [P, A]^x) > 0$ for all x	$[P, A]$ is finite modulo $O_p(G)$. Some open $N \lhd [G, A]$ has $\lambda_p(N) \leq 1$.

Thank you!

(References)

Further results: all primes

Conjugate Sylow subgroups [C]

If G is finite and, for every $p \in \pi(G)$, a Sylow P satisfies

$$\Pr(P, P^x) \geq \epsilon > 0 \qquad \forall x \in G,$$

then G is bounded-by-abelian-by-bounded.

Coprime action [A]

Let A act coprimely. If for every $p \in \pi(G)$ there is an A-invariant Sylow P with

$$\Pr([P, A], [P, A]^x) \geq \epsilon > 0 \qquad \forall x \in G,$$

then $[G, A]$ is bounded-by-abelian-by-bounded.

(back)

Further results: arbitrary p-subgroups

A *component* is a subnormal quasisimple subgroup. Let $E_p(H)$ be the product of the components of H whose orders are divisible by p. Define

$$\frac{O_{p,e_p}(G)}{O_p(G)} = E_p(G/O_p(G)), \qquad \frac{O_{p,e_p,p}(G)}{O_{p,e_p}(G)} = O_p(G/O_{p,e_p}(G)).$$

D.–Guralnick–Morigi–Shumyatsky [C]

If P is any p-subgroup of a finite group G with

$$\Pr(P, P^x) \geq \epsilon > 0 \qquad (x \in G),$$

then

$$|P : P \cap O_{p,e_p,p}(G)| \text{ is } \epsilon\text{-bounded.}$$

Further results: Hall subgroups

D.–Guralnick–Morigi–Shumyatsky [C]

Let $K \leq G$, with G finite, and suppose that

$$\Pr(K, K^x) \geq \epsilon > 0 \qquad (x \in G).$$

- If K is a Hall subgroup, then

$$|K : K \cap F(G)| \text{ is } \epsilon\text{-bounded.}$$

- If every composition factor of Lie type has Lie rank at most n, then

$$|K : K \cap F(G)| \text{ is } (n, \epsilon)\text{-bounded.}$$

The results for p-subgroups apply to the Sylow subgroups of K, since commuting probability increases on passing to subgroups.

A single fixed-point test is insufficient

An example [P]

Let

$$G = \prod_{i=1}^{\infty} \mathrm{PSL}(2, 32),$$

and let α act on every factor as a field automorphism of order 5. Then α is coprime and

$$C_G(\alpha) \cong \prod_{i=1}^{\infty} S_3.$$

The order of $\mathrm{PSL}(2, 32)$ is divisible by 11.

For $T \in \mathrm{Syl}_2(C_G(\alpha))$ and $U \in \mathrm{Syl}_{11}(C_G(\alpha))$,

$$U = 1, \qquad \Pr(T, U) = 1.$$

Yet G is not virtually pro-11-soluble.

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The index need not be bounded

No bound on the index of the prosoluble radical

Let

$$G_r = \mathrm{PSL}(2, r), \qquad r \geq 5 \text{ prime}, \quad r \equiv 3, 5 \pmod 8.$$

A Sylow 2-subgroup L has order 4. For every Sylow subgroup Q,

$$\Pr(L, Q) > 1/4.$$

Yet the soluble radical of G_r is trivial and $|G_r|$ is unbounded.

Thus positivity of $\Pr(S_2, S_3)$ and $\Pr(S_2, S_5)$ can imply virtual prosolubility without bounding the index of the prosoluble radical.

The three papers

Joint work with R. M. Guralnick, M. Morigi and P. Shumyatsky.

[C] Conjugate subgroups

Commuting probability for conjugate subgroups of a finite group.

[A] Finite groups and automorphisms

Finite groups, commuting probability, and coprime automorphisms.

[P] Profinite groups and automorphisms

Coprime automorphisms of profinite groups and the commuting probability of Sylow subgroups.

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