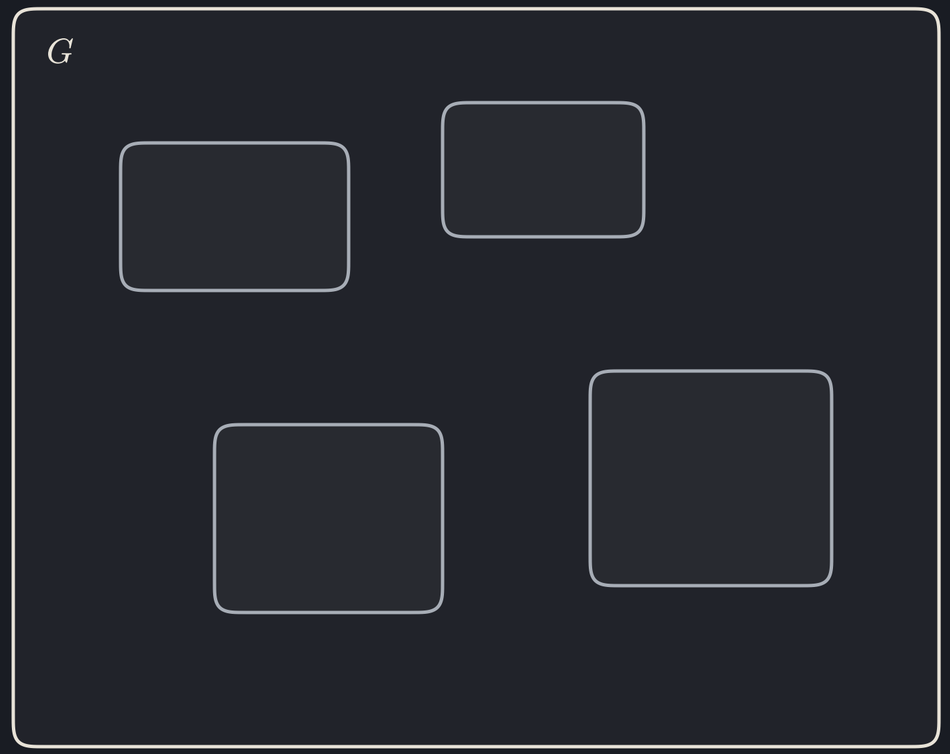
The group class SJ is not countably recognizable

M. Brescia — B.G. Di Siena — E. Ingrosso — M. Trombetti

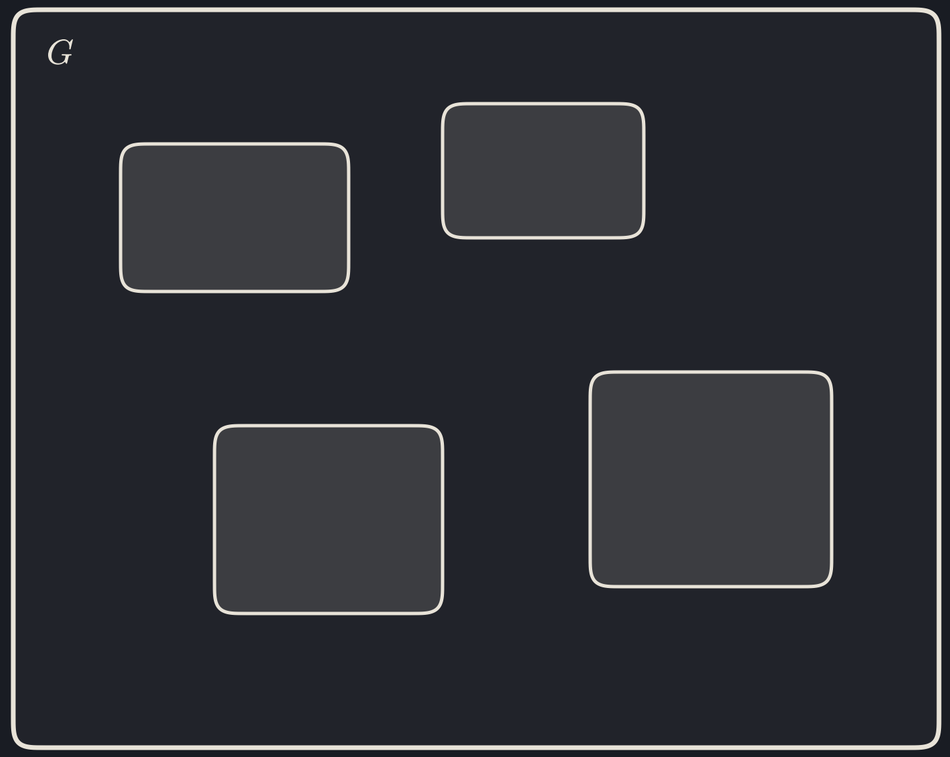


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Countable subgroups $C \leq G$

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Countable subgroups $C \leq G$

$$C \in \mathfrak{X} \text{ for every countable } C \leq G \implies G \in \mathfrak{X}$$

Countable recognizability is therefore a local-to-global principle at the first infinite cardinal

Definition 2.1. A group class $\mathfrak{X}$ is *countably recognizable* if a group belongs to $\mathfrak{X}$ whenever all its countable subgroups are $\mathfrak{X}$-groups.

Detected on countable subgroups

Černikov 1950

Černikov proved that hypercentrality can be detected on countable subgroups

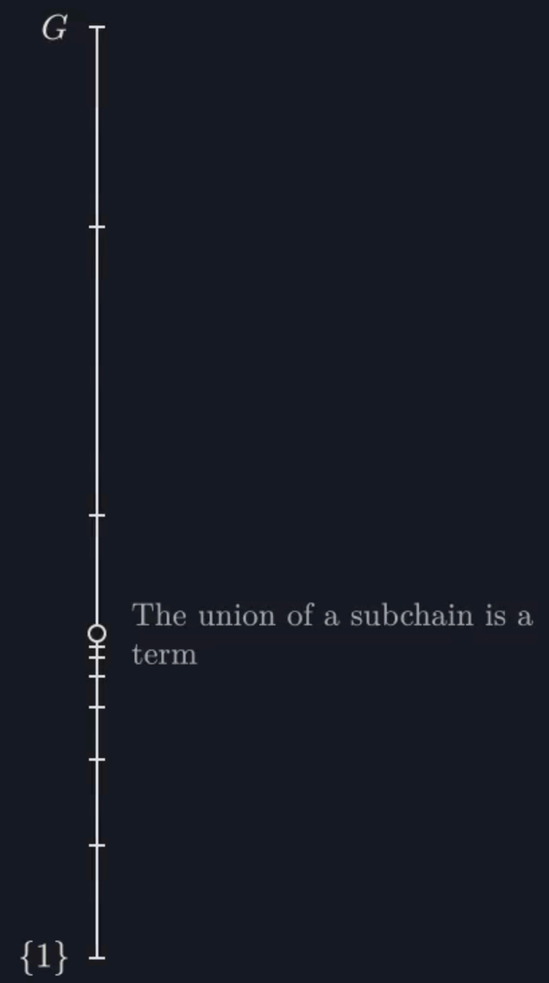
Not detected

Higman 1951

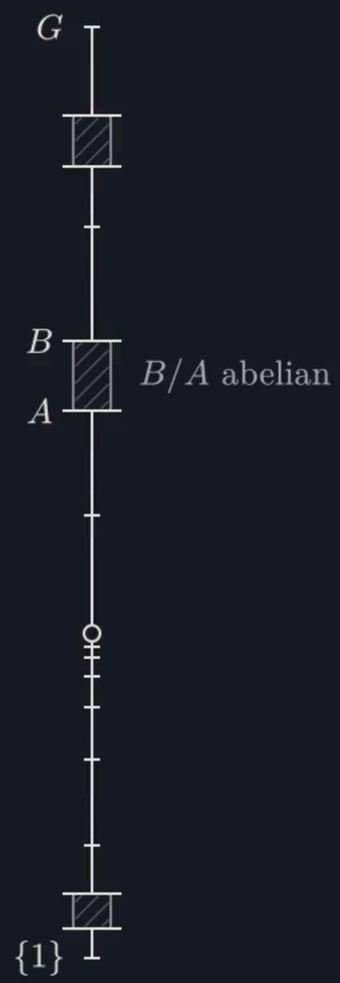
Higman's work on almost free groups yields the classical failure of countable recognizability for free groups

Hickin-Phillips 1970

Hickin and Phillips constructed an uncountable locally finite p-group showing that an ascending-series condition may hold on all countable subgroups and fail globally

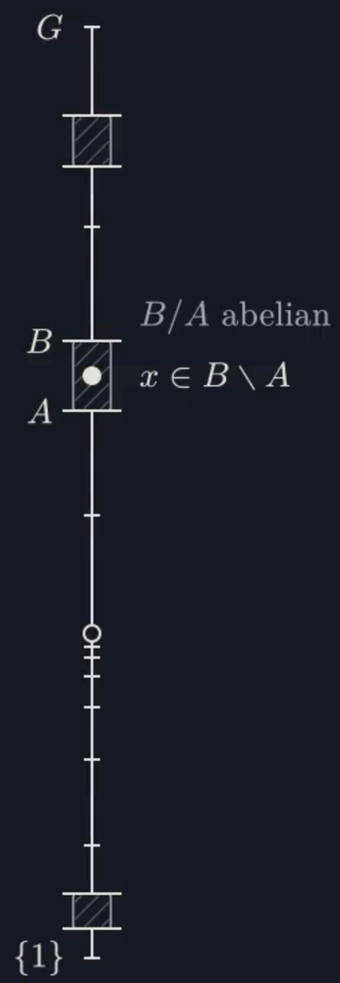


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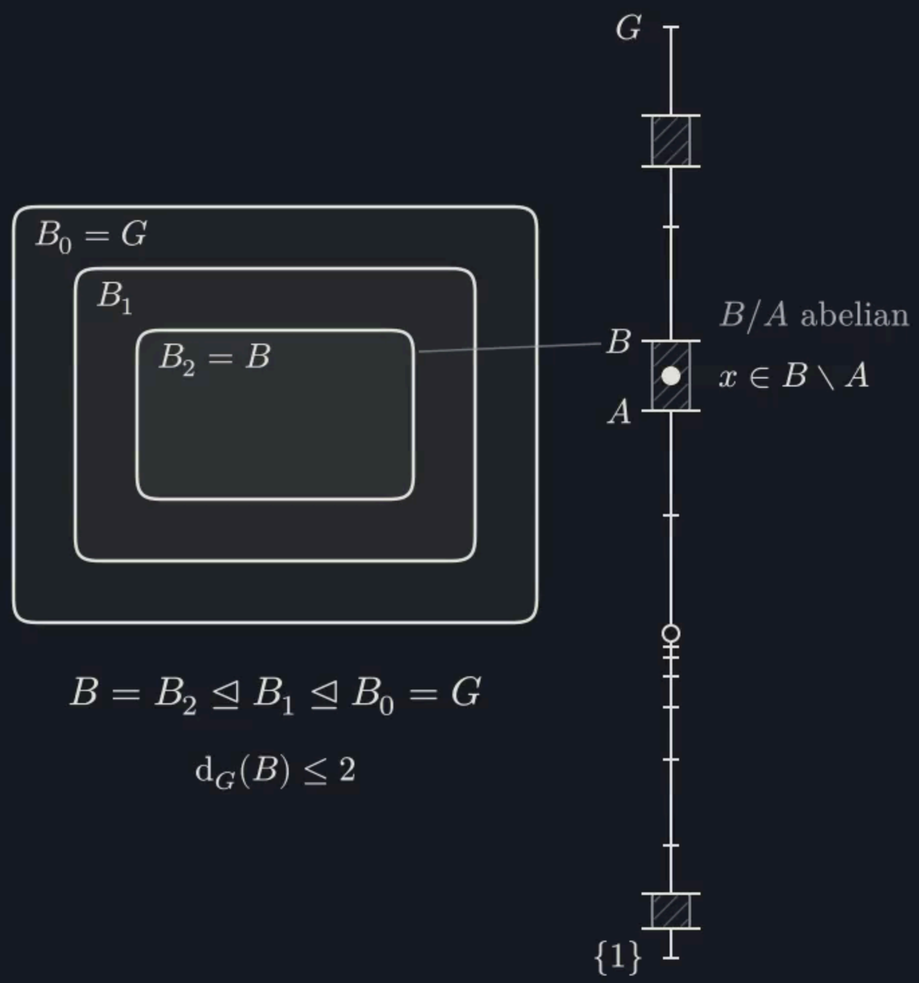


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(i) If $A < B$ are consecutive terms of $\mathcal{S}$, then $A \trianglelefteq B$ and B/A is abelian.

(ii) Every non-identity element of G lies in some $B \setminus A$, for consecutive terms A and B of the series.

We call the couple (A, B) a *jump* and say that an element x of G is *covered by* (A, B) if $x \in B \setminus A$.



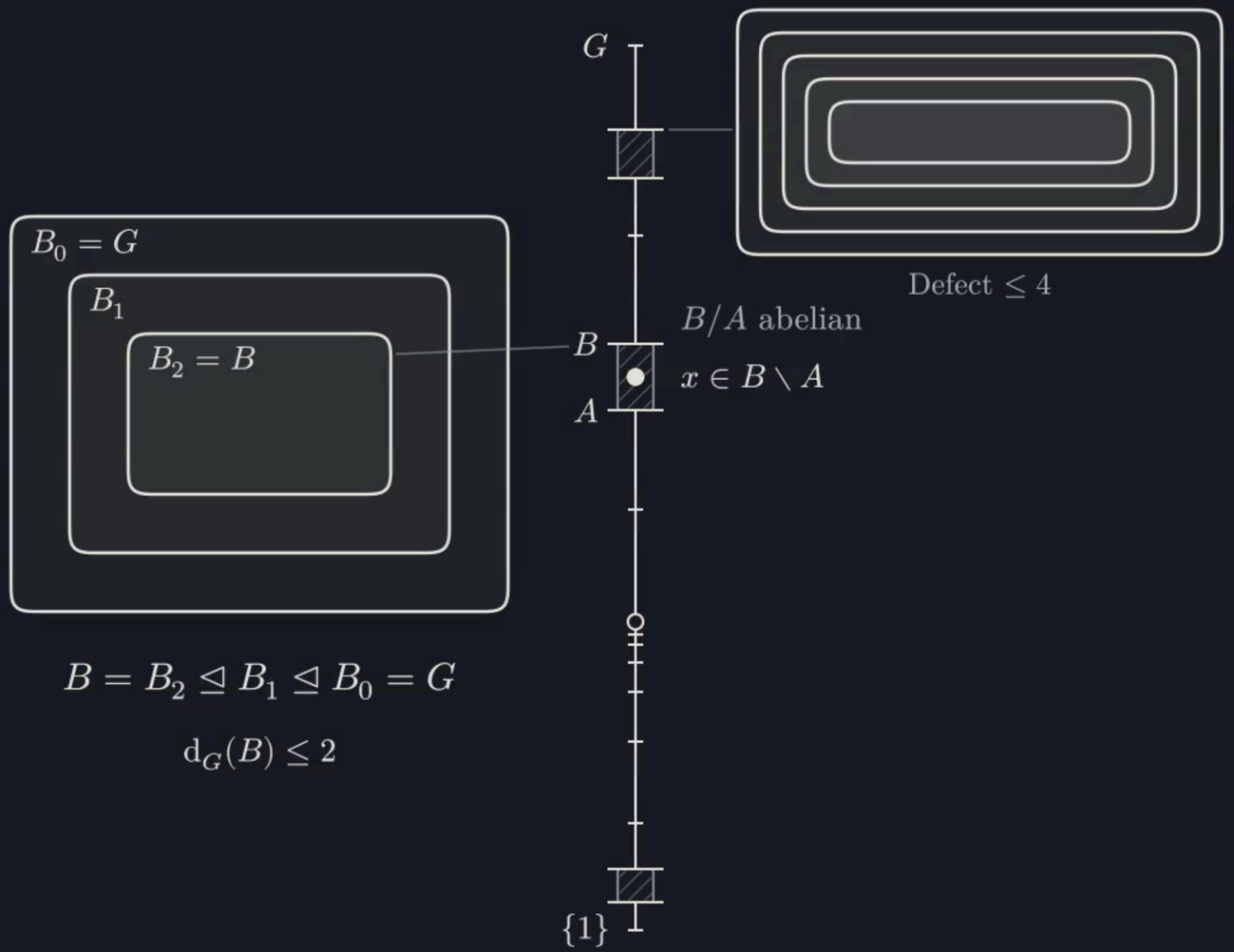
$$B = B_2 \trianglelefteq B_1 \trianglelefteq B_0 = G$$

$$\mathrm{d}_G(B) \leq 2$$

Let $H \leq G$. We write $H \trianglelefteq\trianglelefteq_d G$ if there is a chain

$$H = H_d \trianglelefteq H_{d-1} \trianglelefteq \ldots \trianglelefteq H_0 = G.$$

The least possible d, when it exists, is the (*subnormal*) *defect* of H in G and is denoted by $\mathrm{d}_G(H)$.



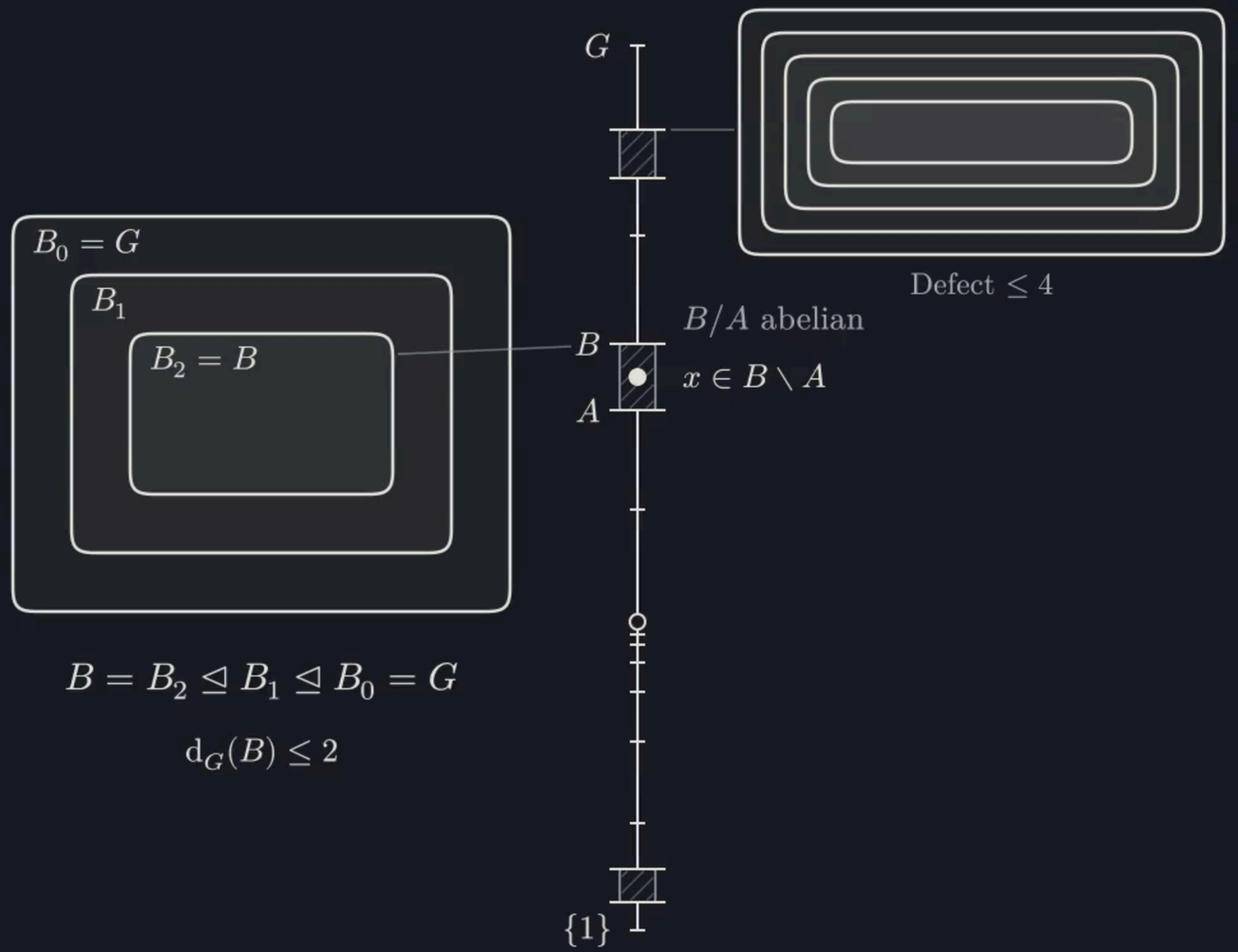
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The group G belongs to SJ if it has a general abelian series every term of which is subnormal of finite defect in G.

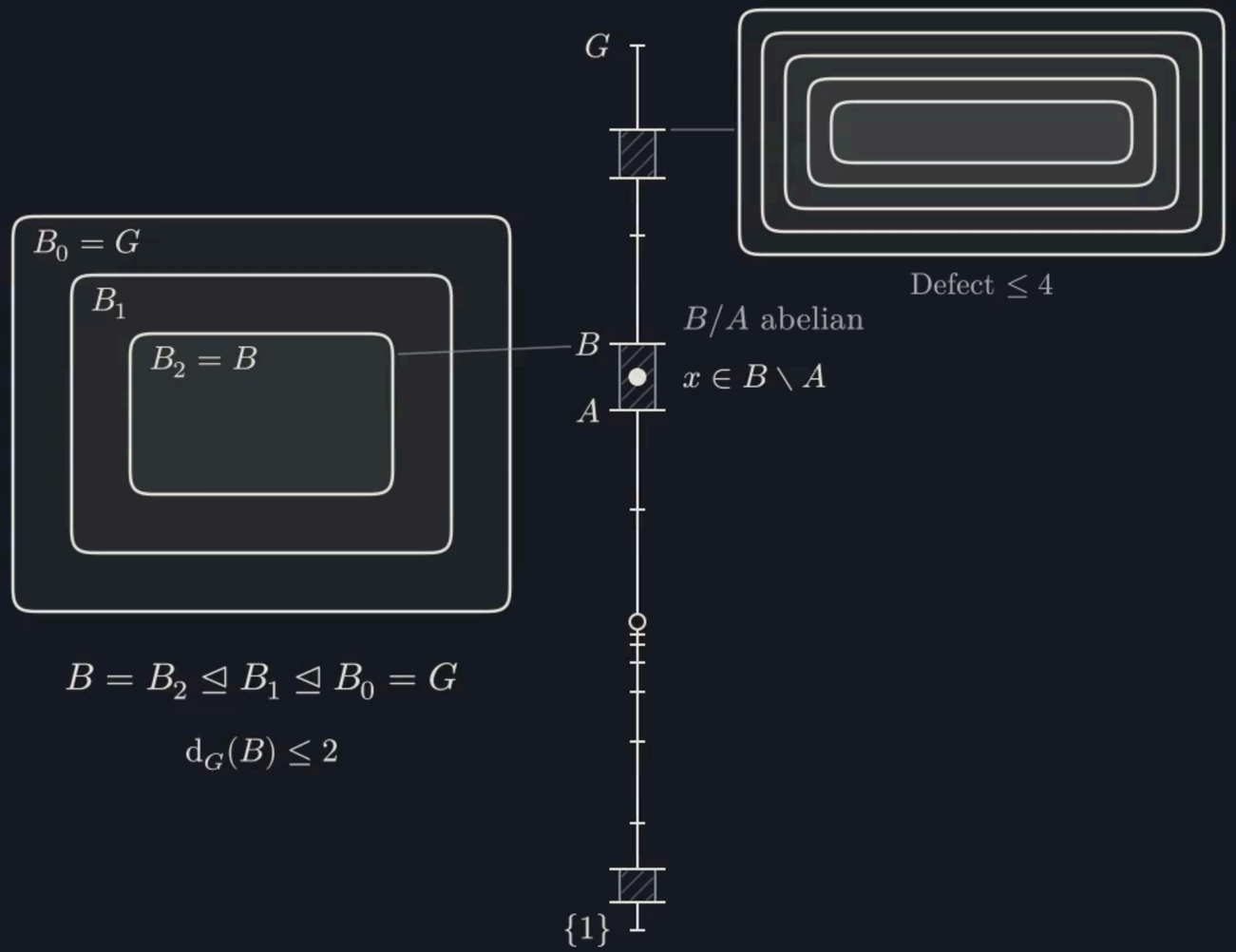


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It is easy to see that every subgroup of an SJ-group is SJ, and that the class of SJ-groups is closed with respect to forming extensions.

However, the class SJ is not closed with respect to forming quotients, as one can see taking into account a free group of rank 2.



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Lemma 2.6. A free product of SJ-groups is an SJ-group.

Consider now the class of SJ-groups.

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We give a negative answer.

Main Theorem. There exists a group G such that every countable subgroup of G belongs to SJ, whereas $G \notin \mathrm{SJ}$. Consequently the class SJ is not countably recognizable.

The factors A_N, $N \geq 1$

F1 · pair→perfect: $\langle x, y\rangle^{A_N, j}$ is perfect for $1 \leq j \leq N$ (Proposition 3.3)

F2 · single elements↛perfect: $x \notin X'$ for $X = \langle x\rangle^{A_N}$, and likewise for y (Lemma 3.4)

F3 · A_N, $\langle x, y\rangle^{A_N}/X$ and $A_N/\langle x, y\rangle^{A_N}$ have finite normal height (Lemma 3.5)

$G \notin \mathrm{SJ}$ (Proposition 5.1)

Uncountably many indices: a pigeonhole on the defects

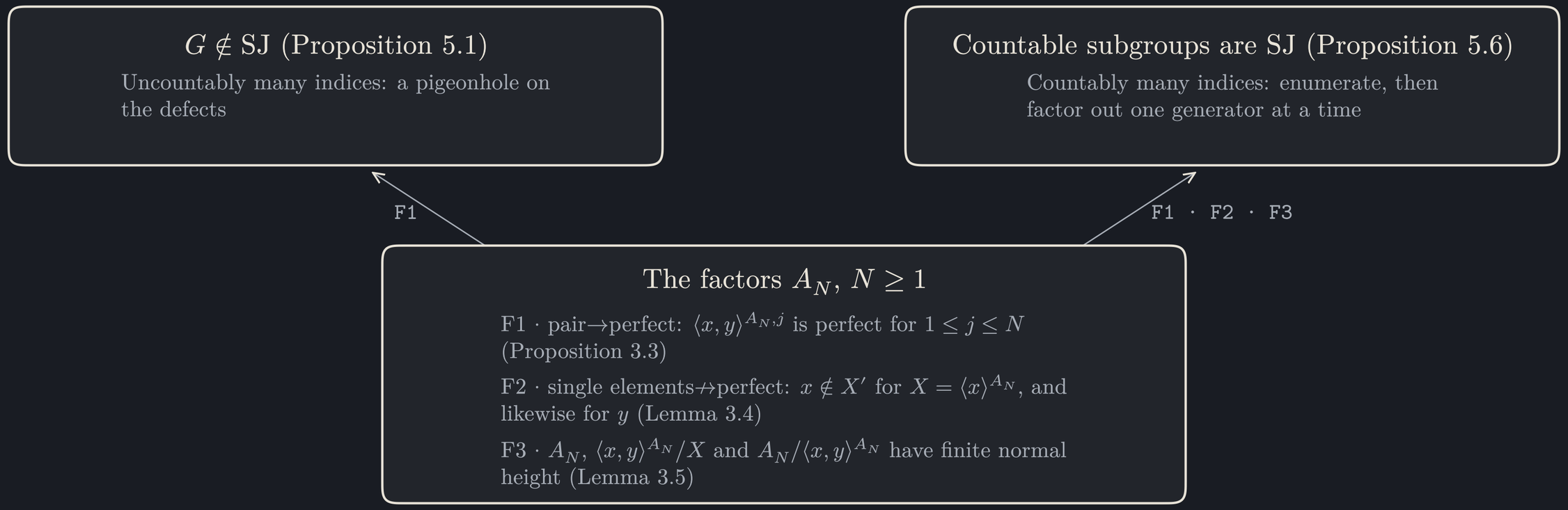
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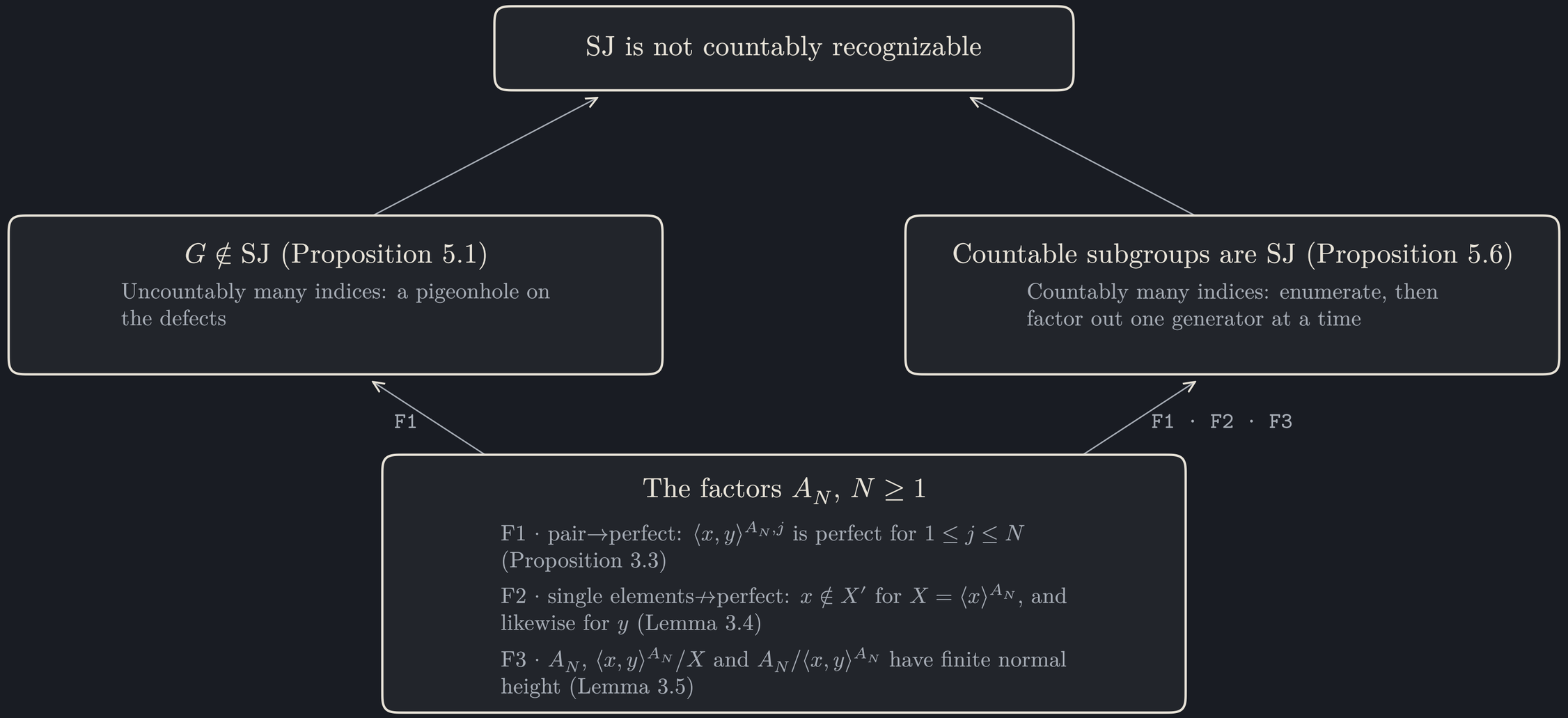
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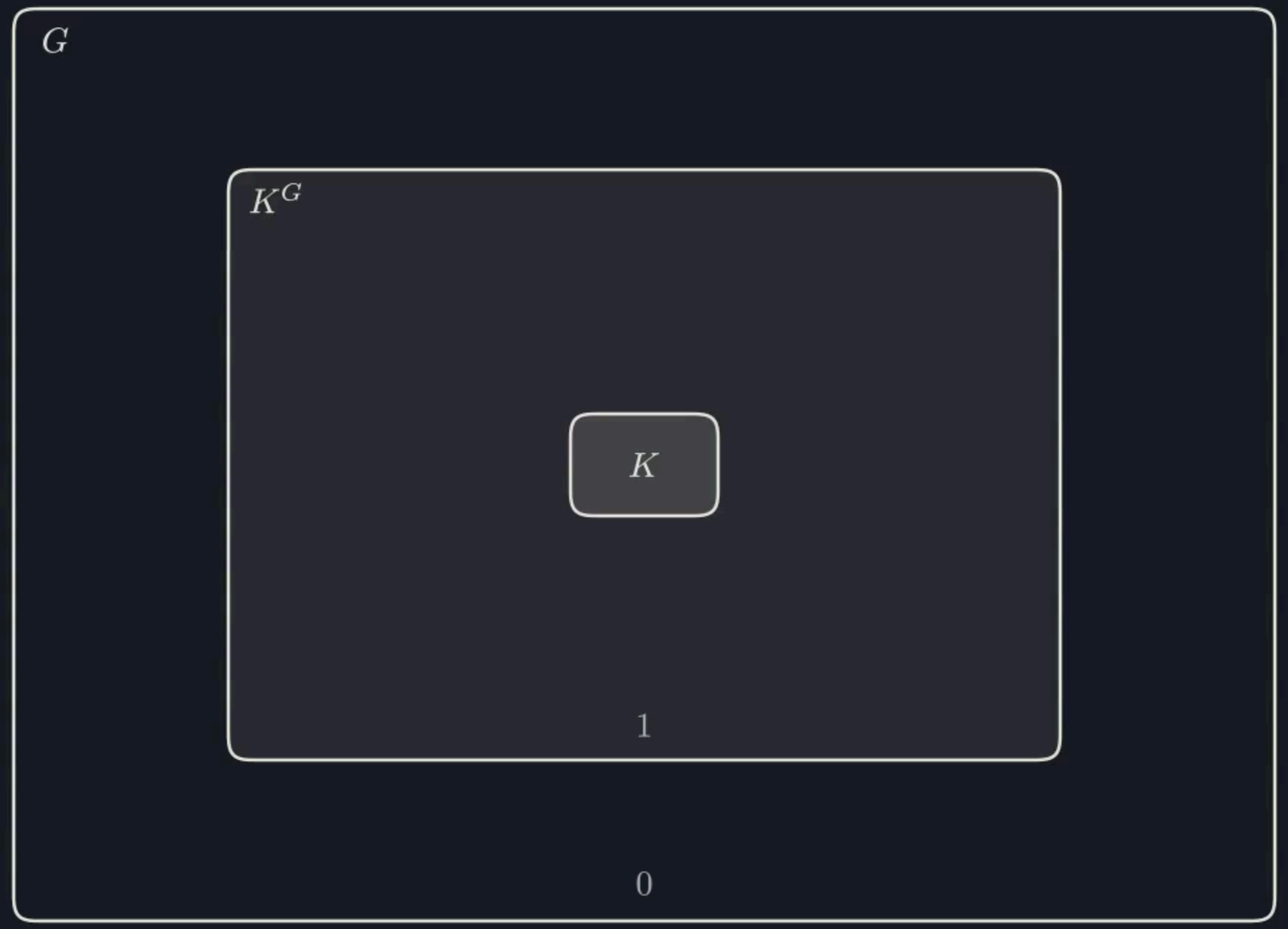
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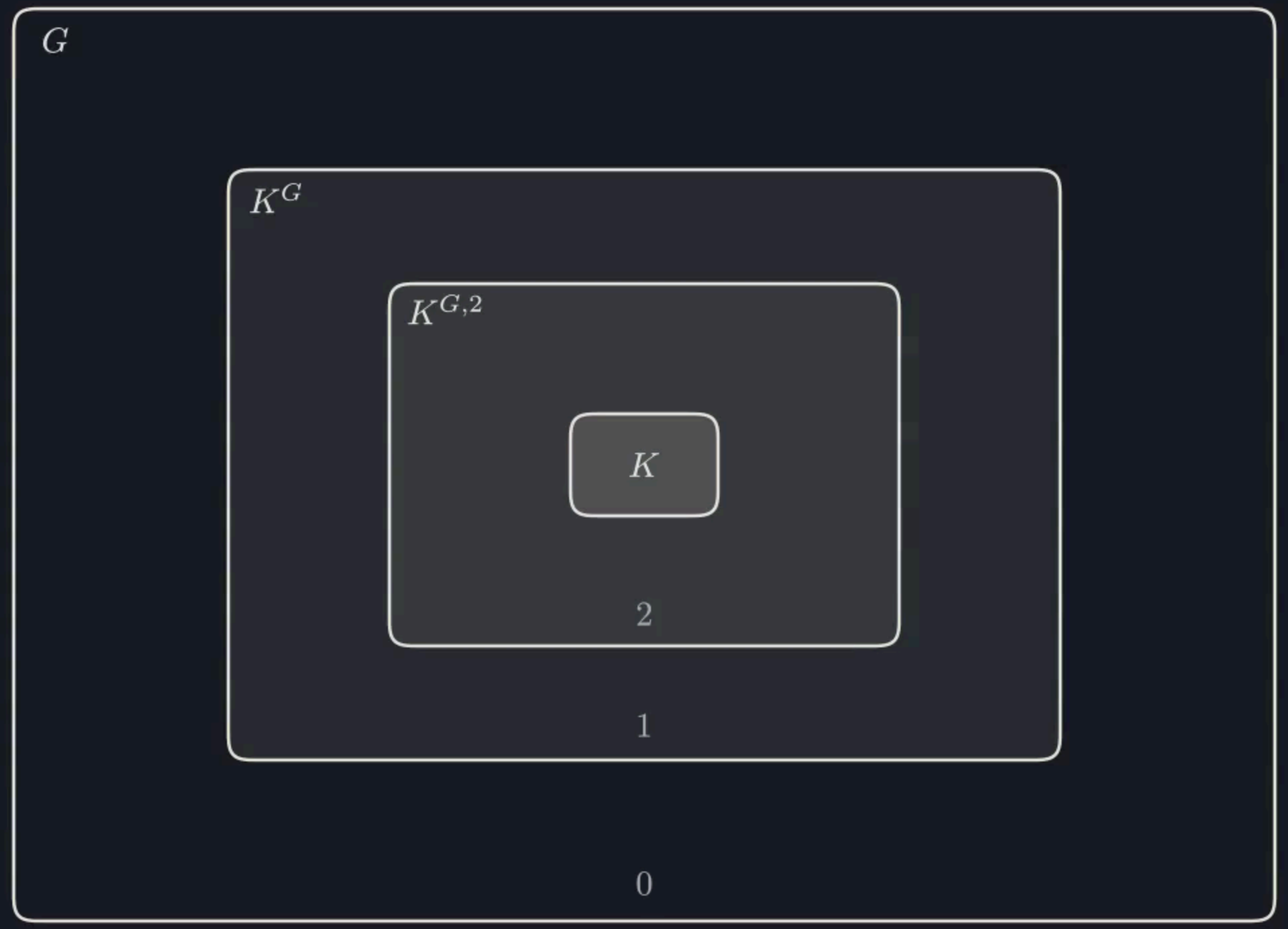
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$$K^{G,0} = G, \qquad K^{G,1} = K^G, \qquad K^{G,n+1} = K^{K^{G,n}}$$





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$K^{G,n}$ is the least subnormal subgroup of G of defect at most n containing K.



Lemma 2.7

Let $K = \langle x, y \rangle \leq G$ and suppose that $K^{G,n}$ is perfect.

Let (A_x, B_x), (A_y, B_y) be the jumps of one general abelian series of G which cover x and y, respectively.

Then B_x and B_y cannot both have defect at most n in G.



$$\mathrm{d}_G(B_x) \le n, \quad \mathrm{d}_G(B_y) \le n$$

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$(A, B) = (A_y, B_y)$

y

x

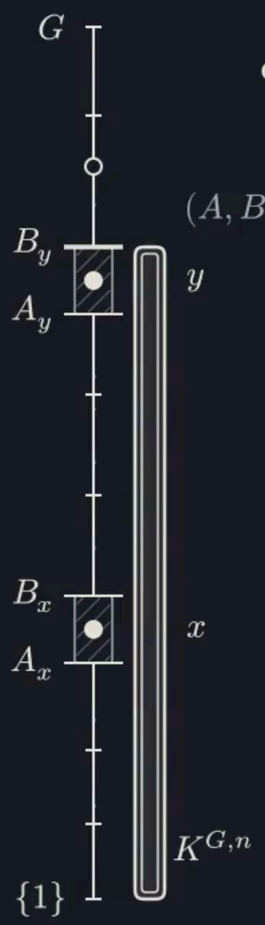
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Proof. Assume the contrary and let (A, B) be the higher of the two jumps.



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$$K \le B \implies K^{G,n} \le B$$

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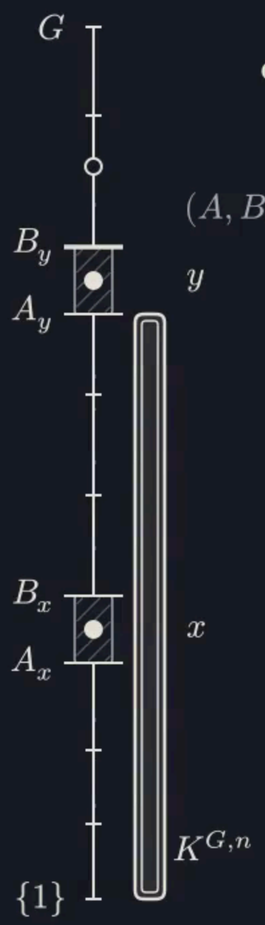
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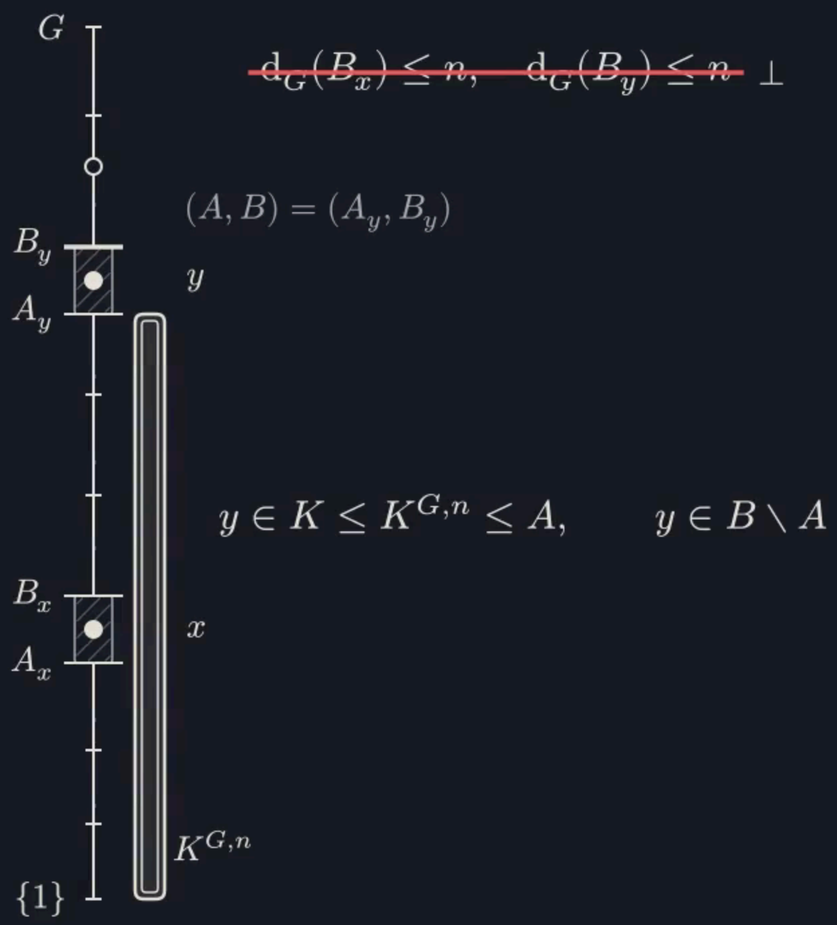
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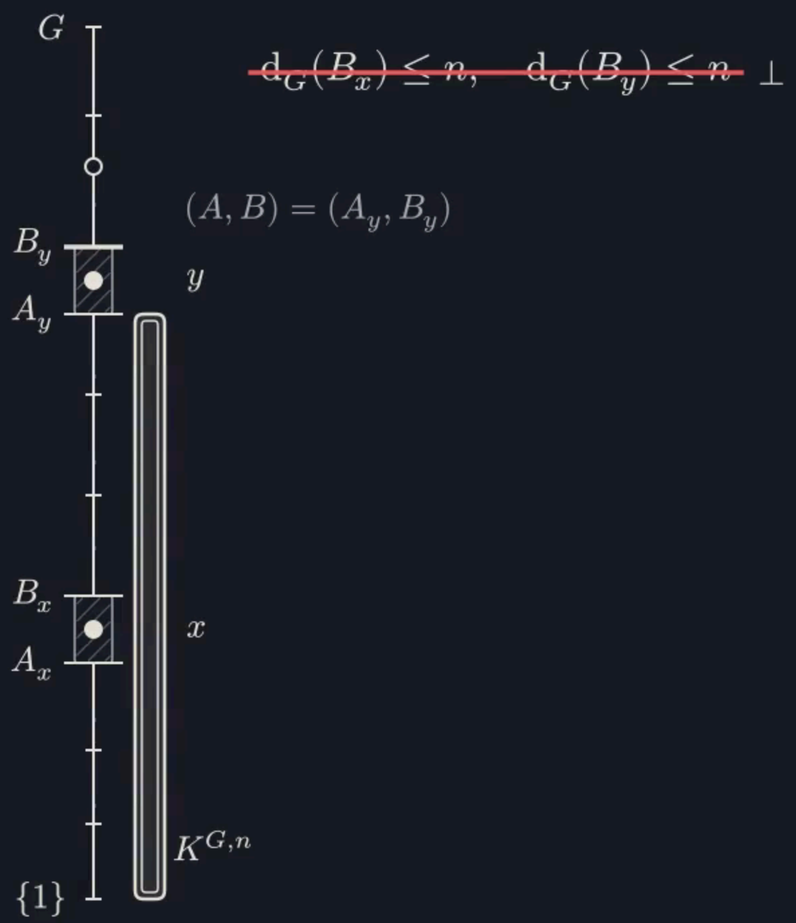
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Then $K \leq B$, so also $K^{G,n} \leq B$.

Since $K^{G,n}$ is perfect, so $K^{G,n} \leq B' \leq A$.

But the element covered by (A, B) belongs to $K \leq K^{G,n}$, contradicting the fact that it lies outside A. $\square$



Perfect subgroups do occur in SJ-groups: there are non-trivial perfect groups of finite normal height (Theorem 2.8(iii)), and these are SJ (Lemma 2.9). The contradiction needs the trap and the covered element together.

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$$K \le A \le B$$

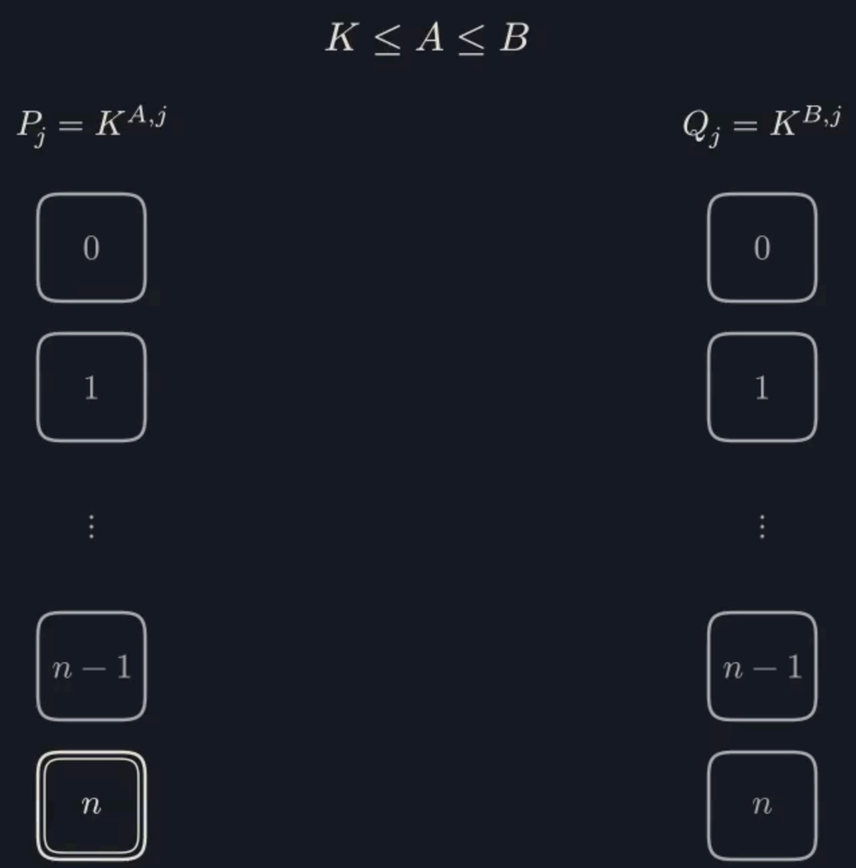
$P_j = K^{A,j}$



Lemma 2.4

Let $K \le A \le B$ be groups, and let $n \ge 1$. If $K^{A,n}$ is perfect, then $K^{B,n}$ is perfect.

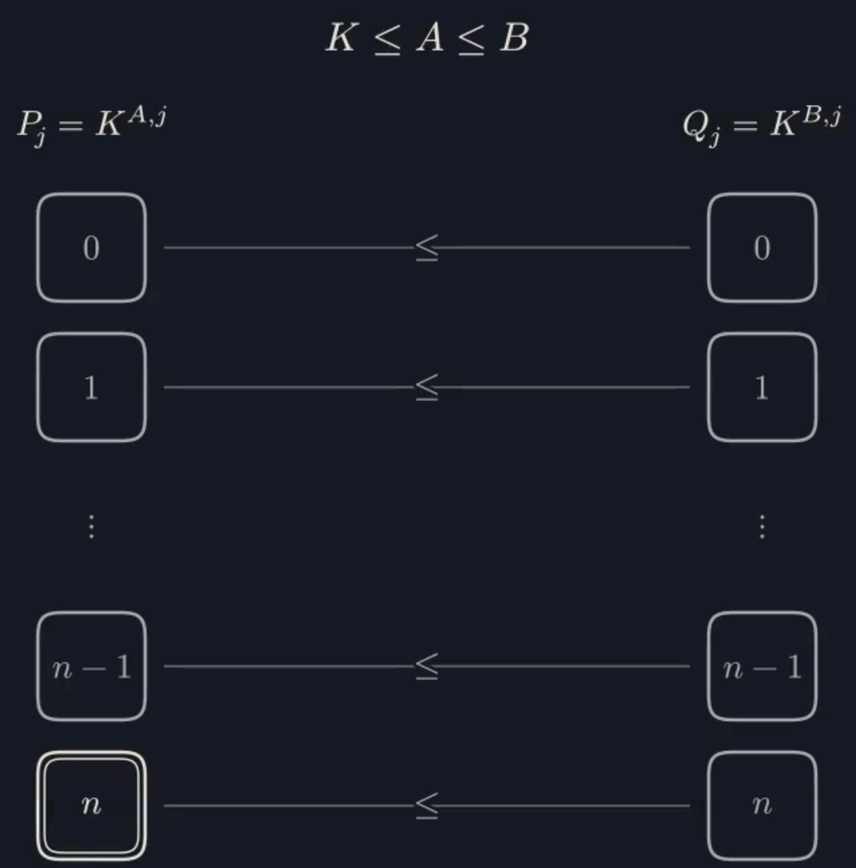
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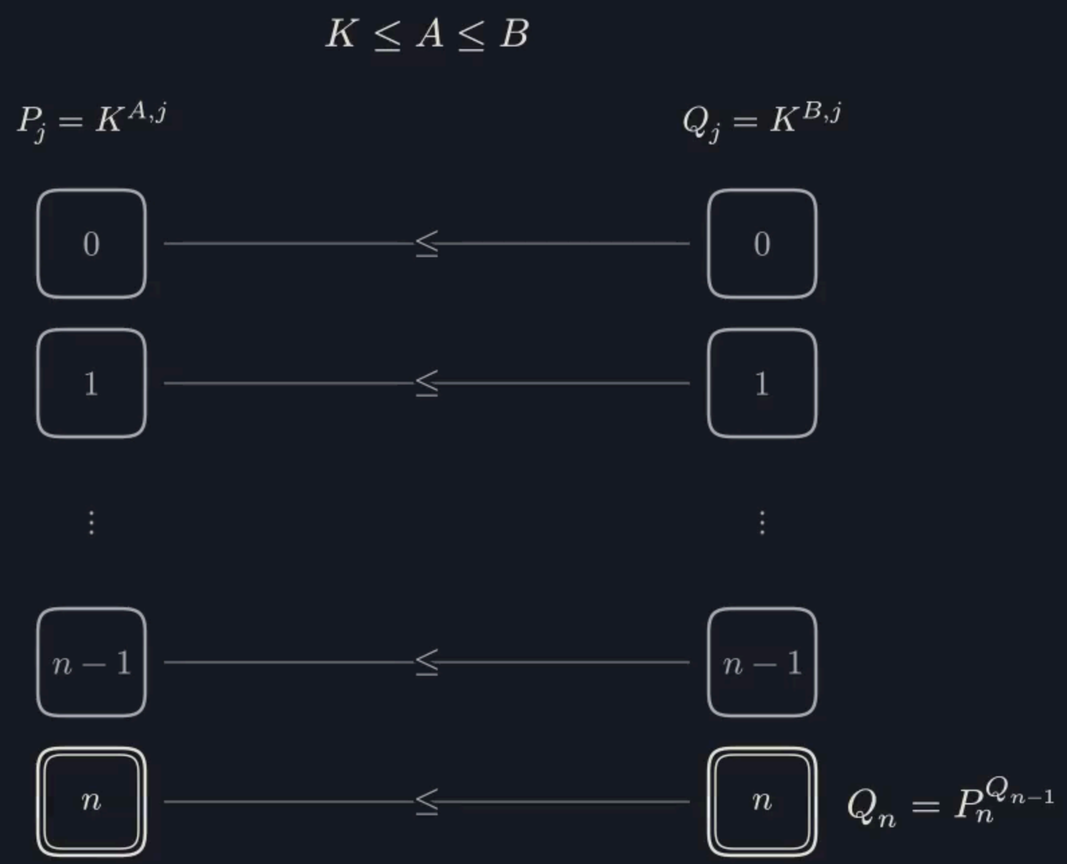
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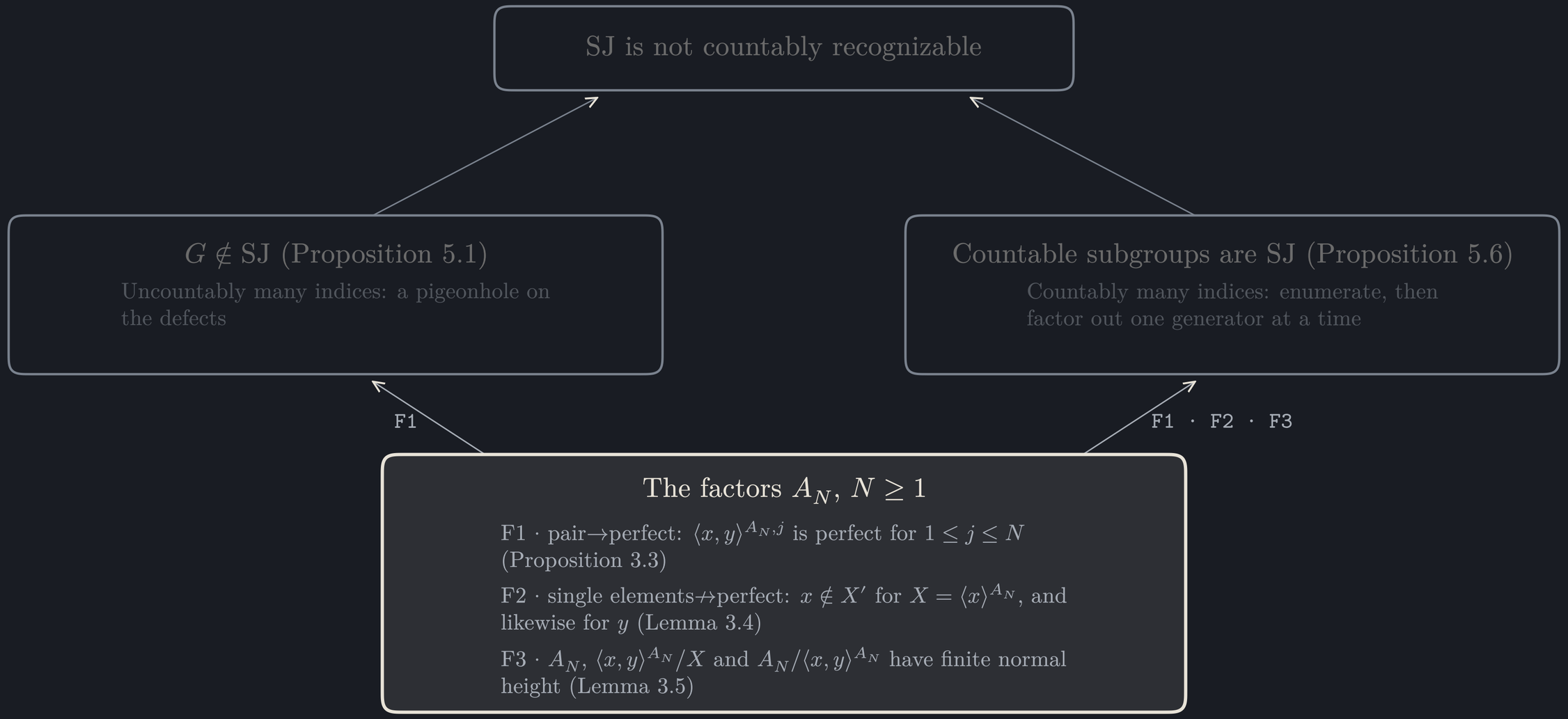
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We proved that $Q_n = P_n^{Q_{n-1}}$.

However, since P_n is perfect, $P_n^{Q_{n-1}} \le Q_n'$. Hence $Q_n = Q_n'$, so $K^{B,n}$ is perfect. $\square$


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Put $\mathcal{H}_0$ equal to the class of abelian groups. Inductively, let $\mathcal{H}_{h+1}$ consist of the groups G which possess a complete linearly ordered system of normal subgroups whose factors belong to $\mathcal{H}_h$. We say that G has *finite normal height* if $G \in \mathcal{H}_h$ for some $h < \omega$.

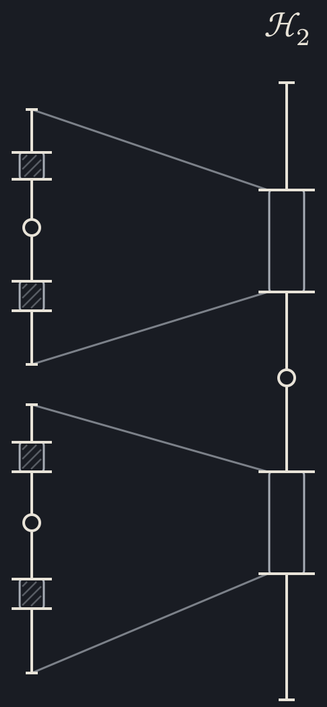


A complete normal system, abelian factors

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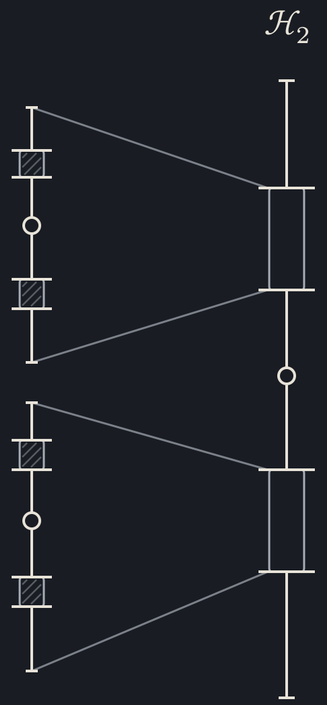


A complete normal system, factors in $\mathcal{H}_1$

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A complete normal system, abelian factors



A complete normal system, factors in $\mathcal{H}_1$

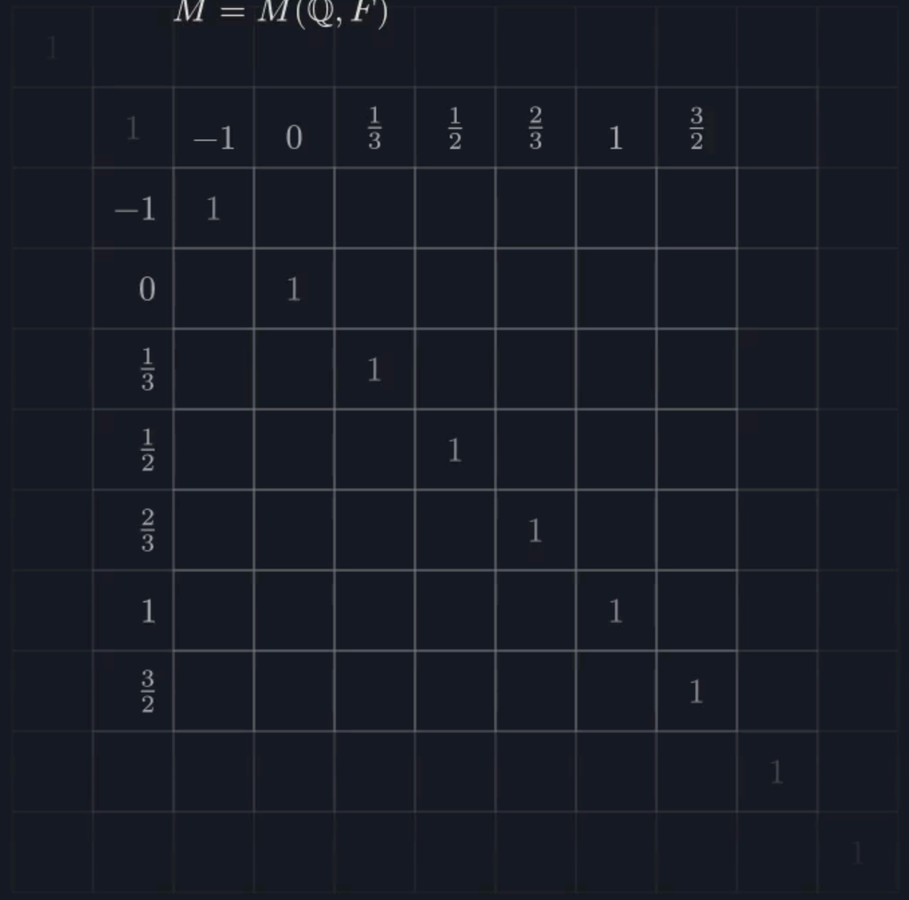
Lemma 2.9. Every group of finite normal height belongs to SJ.

A term of defect d in a factor lifts to a subgroup of defect at most $d + 1$ in G.

One more layer: $T(H)$

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$M = M(\mathbb{Q}, F)$



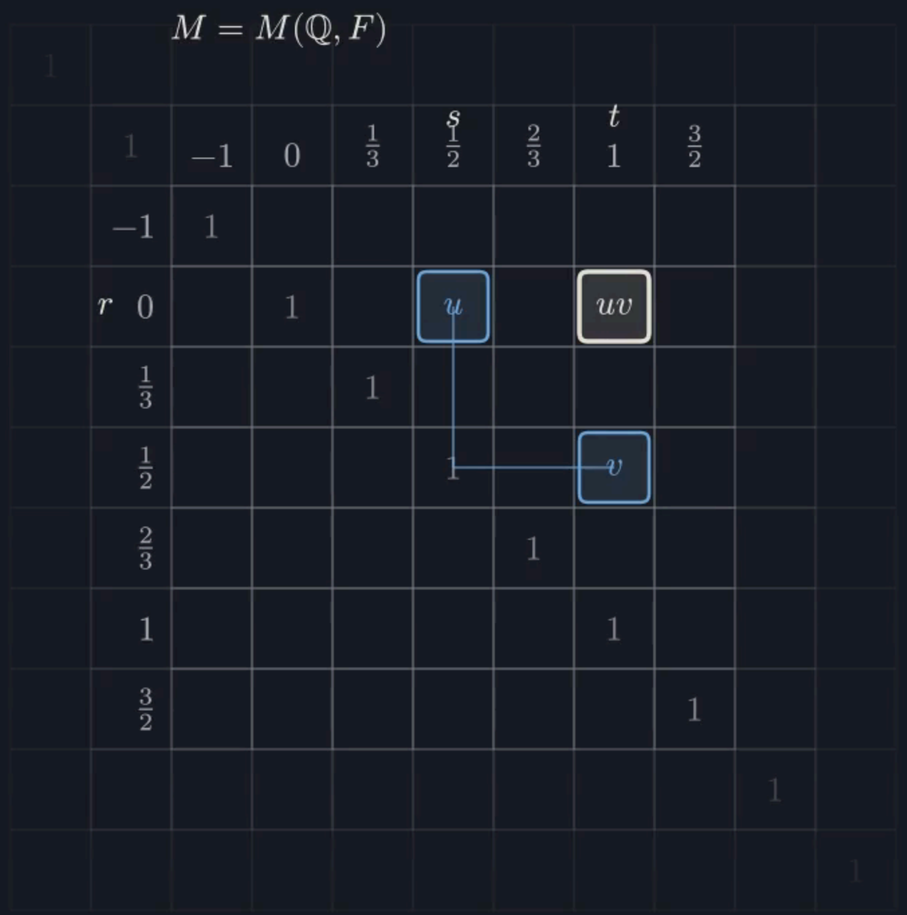
A window of the $\mathbb{Q} \times \mathbb{Q}$ array: M is not a matrix group of finite size

Definition: Robinson, p. 14

Let F be a field of characteristic zero, and let $M = M(\mathbb{Q}, F)$ be the corresponding McLain group

$$x_{rs}(u) = 1 + u\, e_{rs} \qquad (r < s \text{ in } \mathbb{Q},\ u \in F)$$

One more layer: $T(H)$



$F = \mathbb{Q}$ here only as an illustration: any field of characteristic zero

Definition: Robinson, p. 14

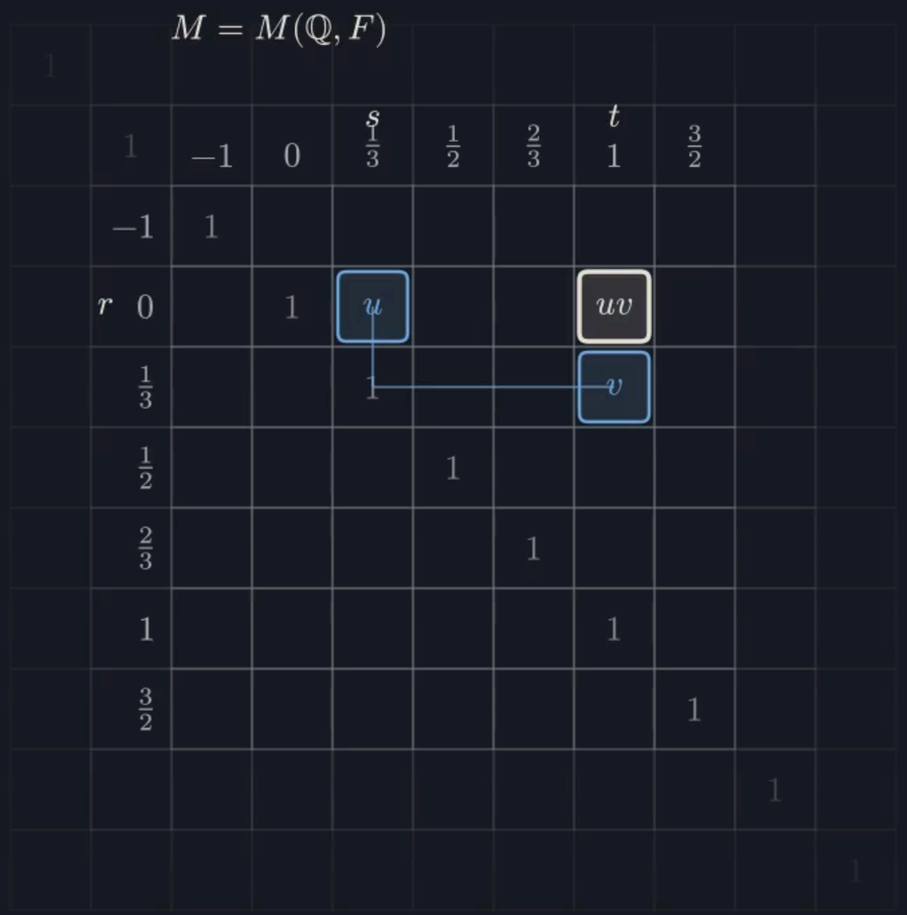
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$$[x_{0,\frac{1}{2}}(1),\ x_{\frac{1}{2},1}(1)] = x_{0,1}(1)$$

One more layer: $T(H)$



Any s with $r < s < t$ will do: $\mathbb{Q}$ is dense, so every generator is a commutator

Definition: Robinson, p. 14

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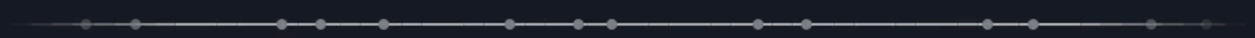
$$[x_{0,\frac{1}{2}}(1),\ x_{\frac{1}{2},1}(1)] = x_{0,1}(1)$$

The group M is non-trivial, perfect, torsion-free and locally nilpotent.

One more layer: $T(H)$

Moreover, every torsion-free locally nilpotent group is right-orderable, so we may fix a right-invariant total order on M. Let Λ denote the set M endowed with this order.

$$\Lambda = (M, <)$$



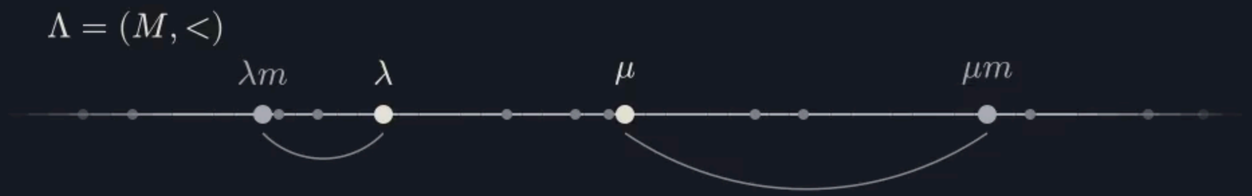
The points of Λ are elements of M, not the rational indices of its matrices; positions are only an order

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Then the action of M on Λ by right multiplication is transitive and order-preserving.

$$\rho_m : \lambda \longmapsto \lambda m$$

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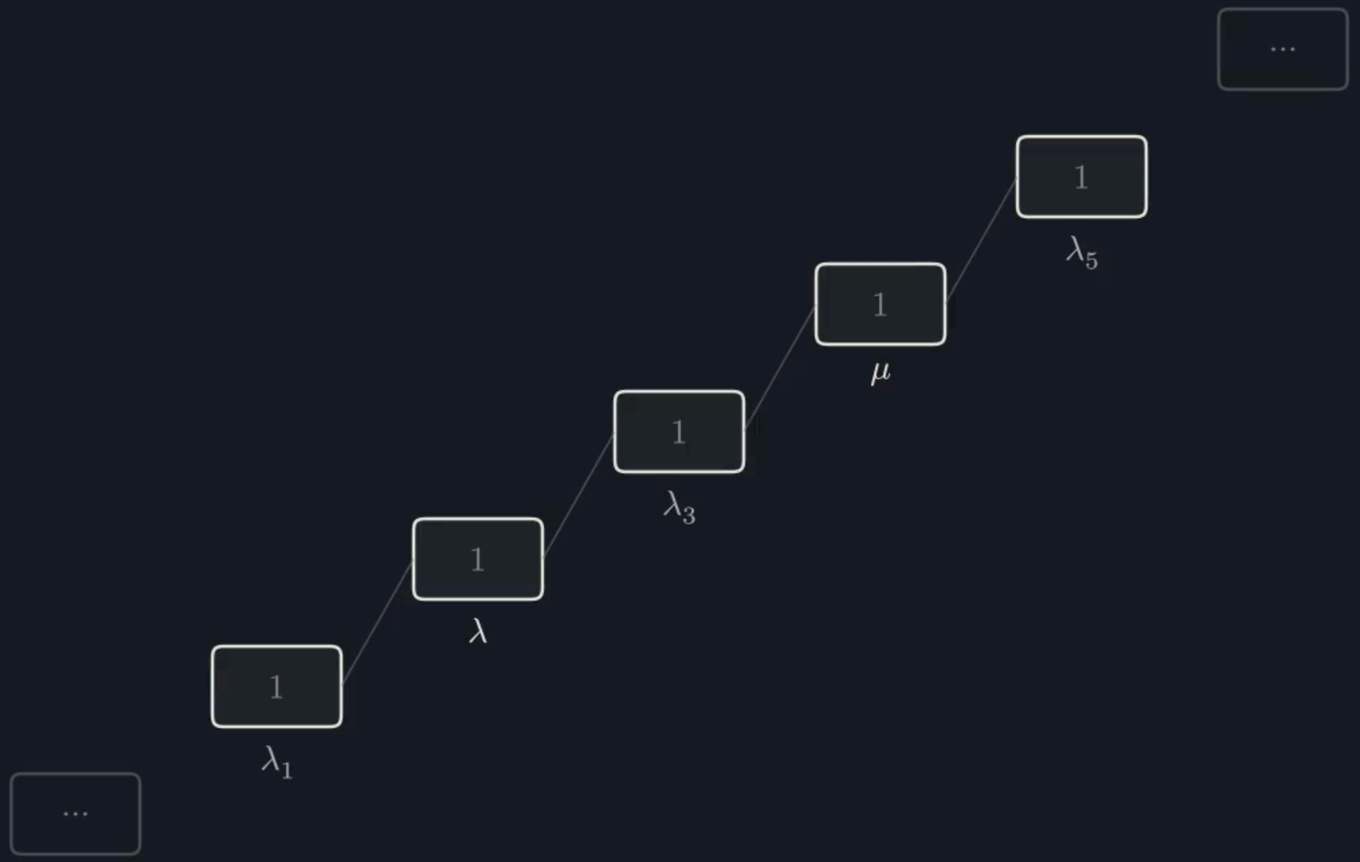
$\Lambda = (M, <)$



$$\cdots < m^{-1} < 1 < m < m^2 < \cdots \qquad (m > 1)$$

One orbit, of one element $m > 1$: the whole group M acts transitively on Λ

One more layer: $T(H)$



Construction: P. Hall (1962)

Now, let H be a perfect group of normal height $h \geq 1$ (just as $M(\mathbb{Q}, \mathbb{Q})$), and form the wreath power $W = \mathrm{Wr}_\Lambda H$.

A state is a function $\varphi : \Lambda \to H$ with $\varphi(\lambda) = 1$ for almost all λ.

W permutes the states: for each $\lambda \in \Lambda$, a copy $H_\lambda = \{h_\lambda : h \in H\}$ of H acts on the λ-coordinate, and $W = \langle H_\lambda : \lambda \in \Lambda \rangle$.

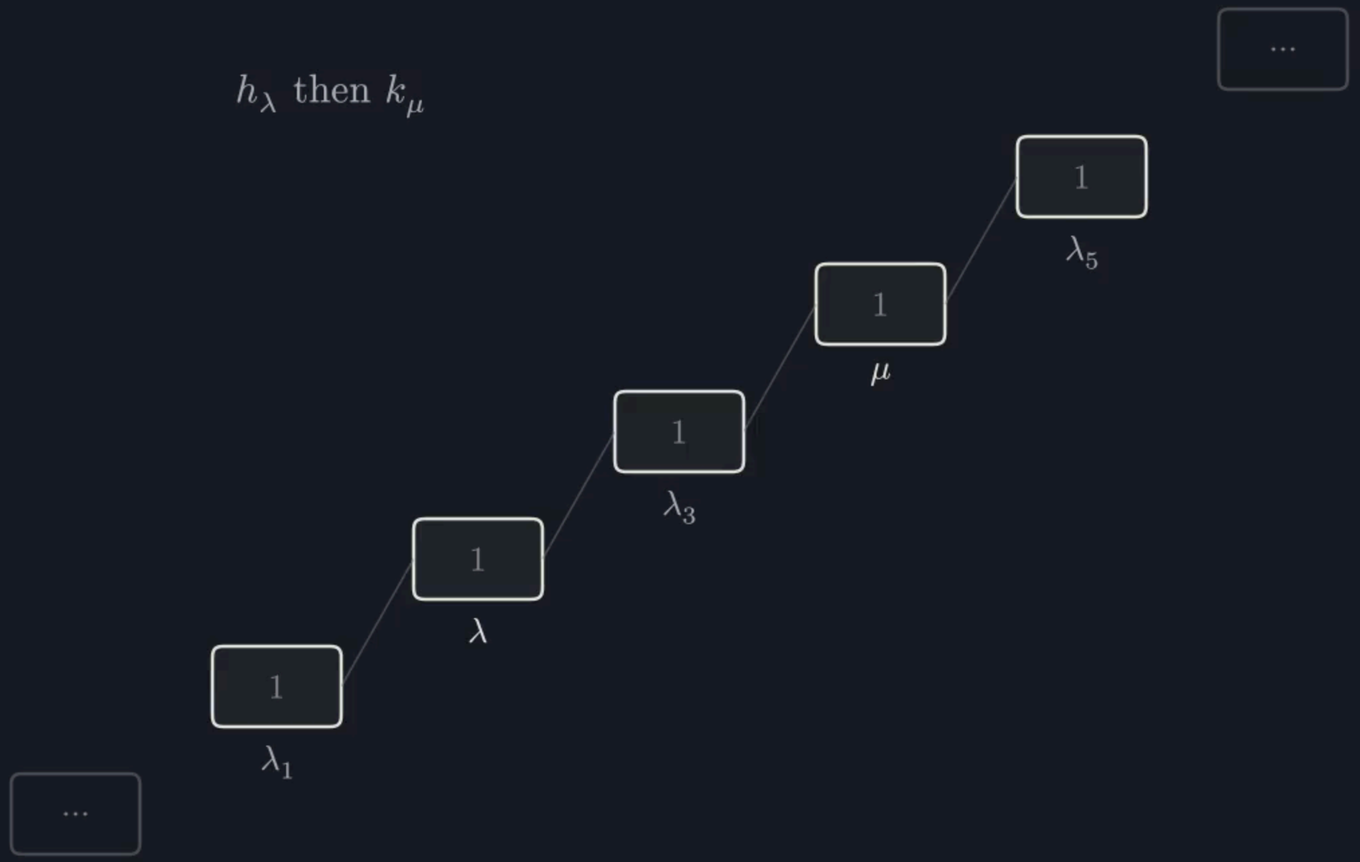
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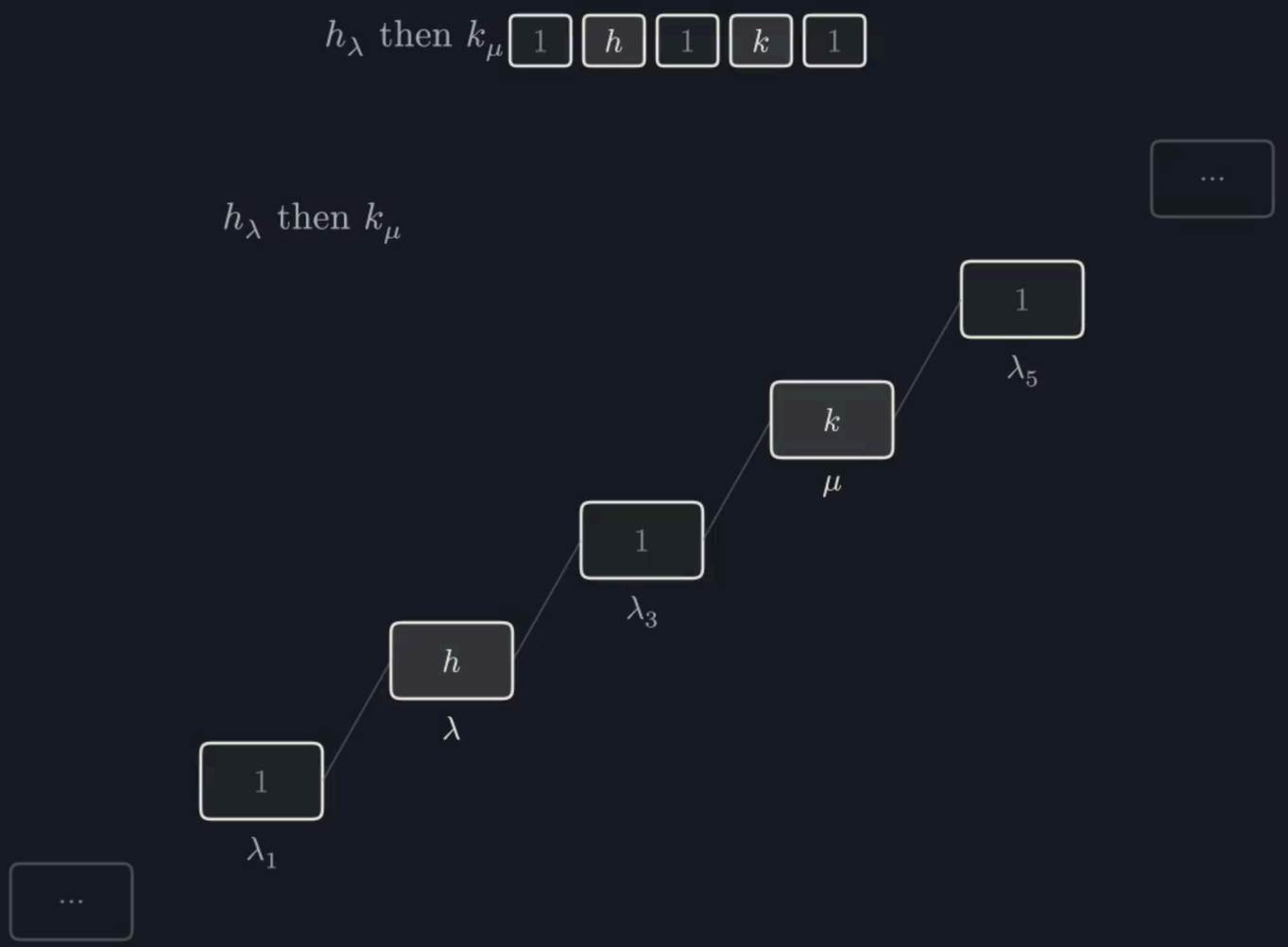
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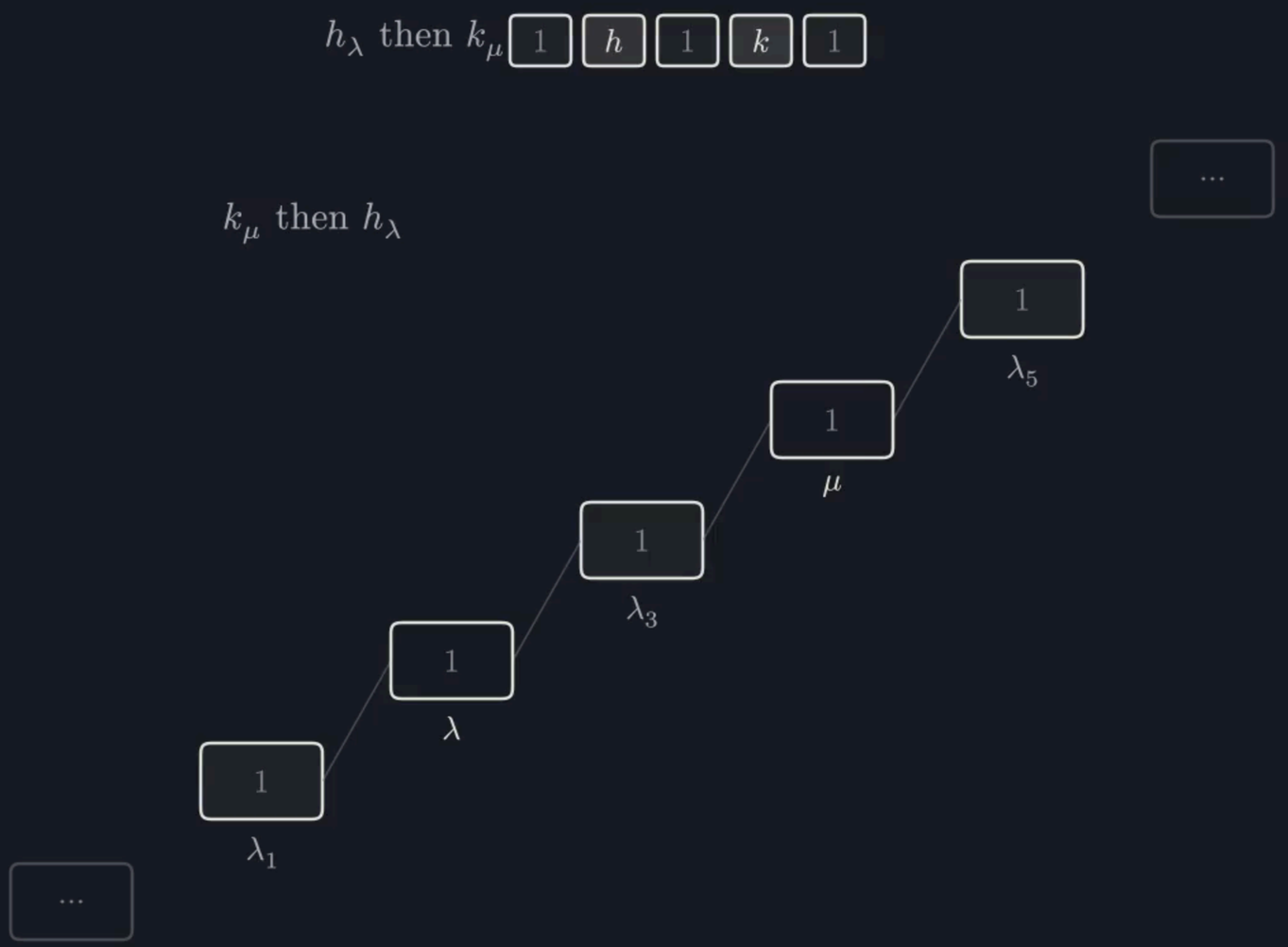
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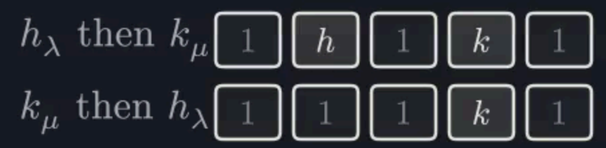
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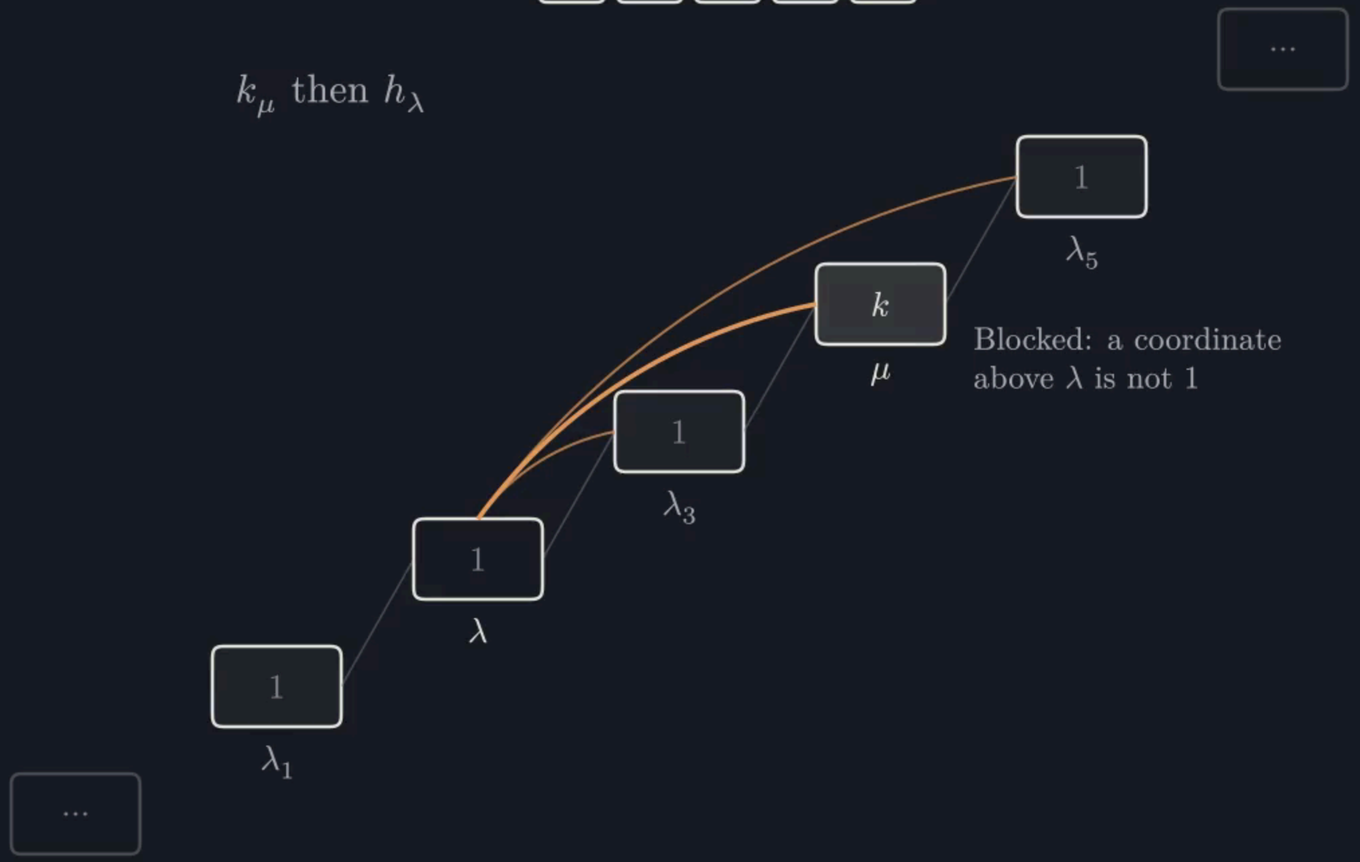
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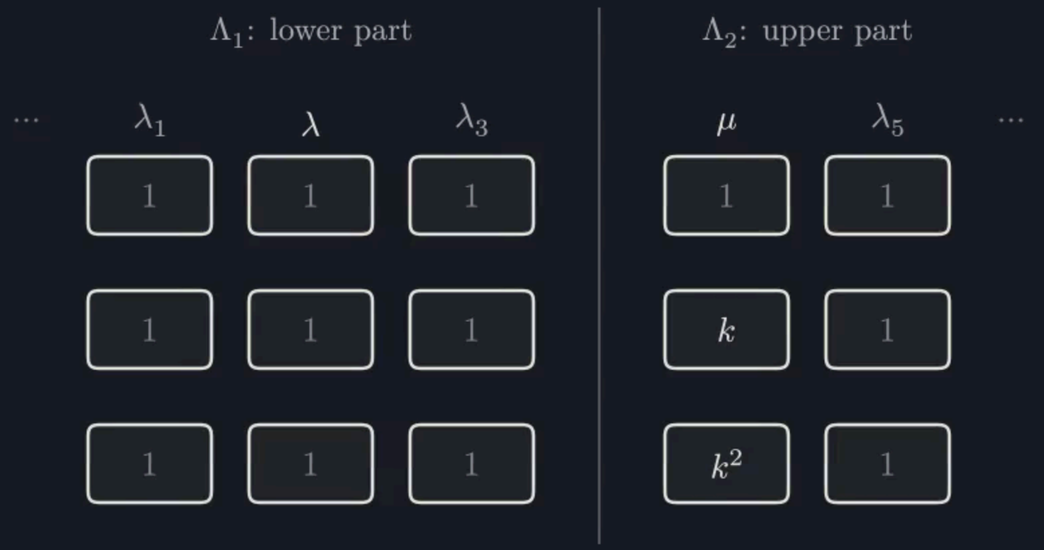
h_λ multiplies the λ-coordinate by h when every coordinate above λ is 1, and does nothing otherwise.

Different states: h_λ and k_μ do not commute. W is not a direct product of the H_λ, nor the wreath product $H \wr M$.

One more layer: $T(H)$

$$\mathrm{Wr}_{\Lambda}\, H \cong \mathrm{Wr}_{\Lambda_1}\, H \wr \mathrm{Wr}_{\Lambda_2}\, H \qquad (\Lambda_1 < \Lambda_2)$$

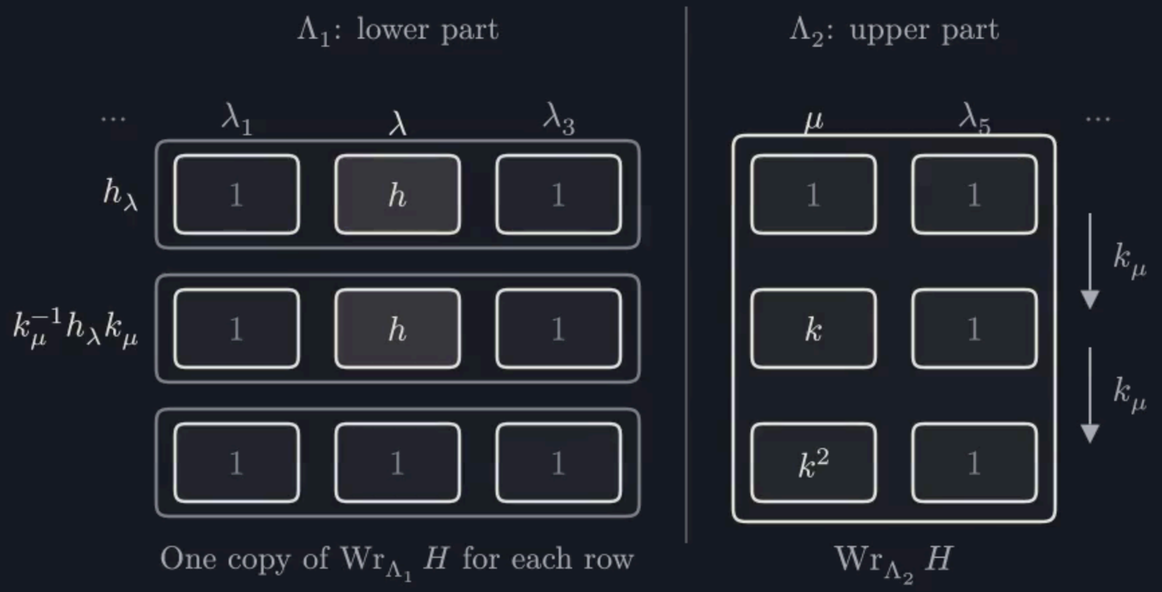
Hall's segmentation law: cut Λ into a lower part Λ_1 and an upper part Λ_2, and sort the states by their upper part, one row for each.



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Hall's segmentation law: cut Λ into a lower part Λ_1 and an upper part Λ_2, and sort the states by their upper part, one row for each.

The upper generators look only at the upper part: $\mathrm{Wr}_{\Lambda_2} H$ permutes the rows.

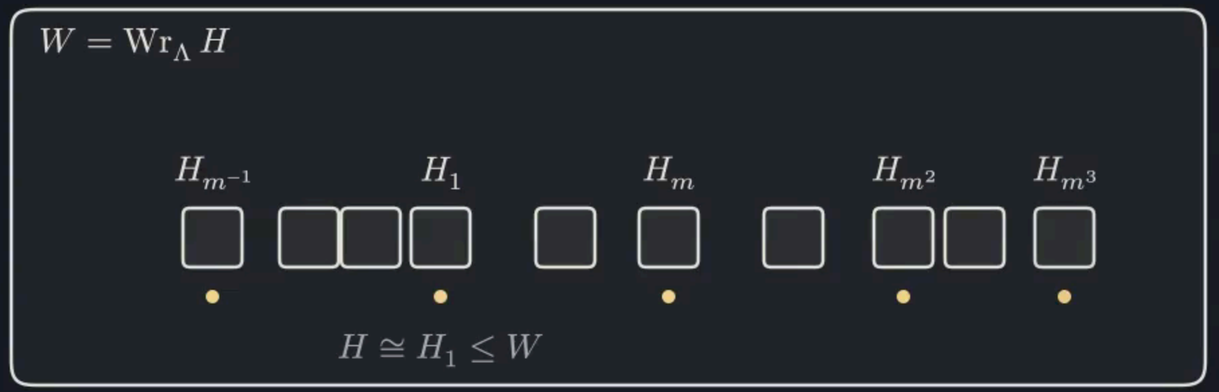
A lower generator h_λ acts only where the upper part is 1; its conjugate by k_μ acts only on the row of k. So each row has its own copy of $\mathrm{Wr}_{\Lambda_1} H$.

Cutting at every point: W has a normal series whose factors are direct products of copies of H, so every such factor has normal height at most h, and hence $W \in \mathcal{H}_{h+1}$.

Construction: P. Hall (1962)

One more layer: $T(H)$

$$W = \langle H_\lambda : \lambda \in \Lambda \rangle, \qquad H_\lambda = \{h_\lambda : h \in H\} \cong H$$



A finite window of coordinates: Λ is infinite

Construction: P. Hall (1962)

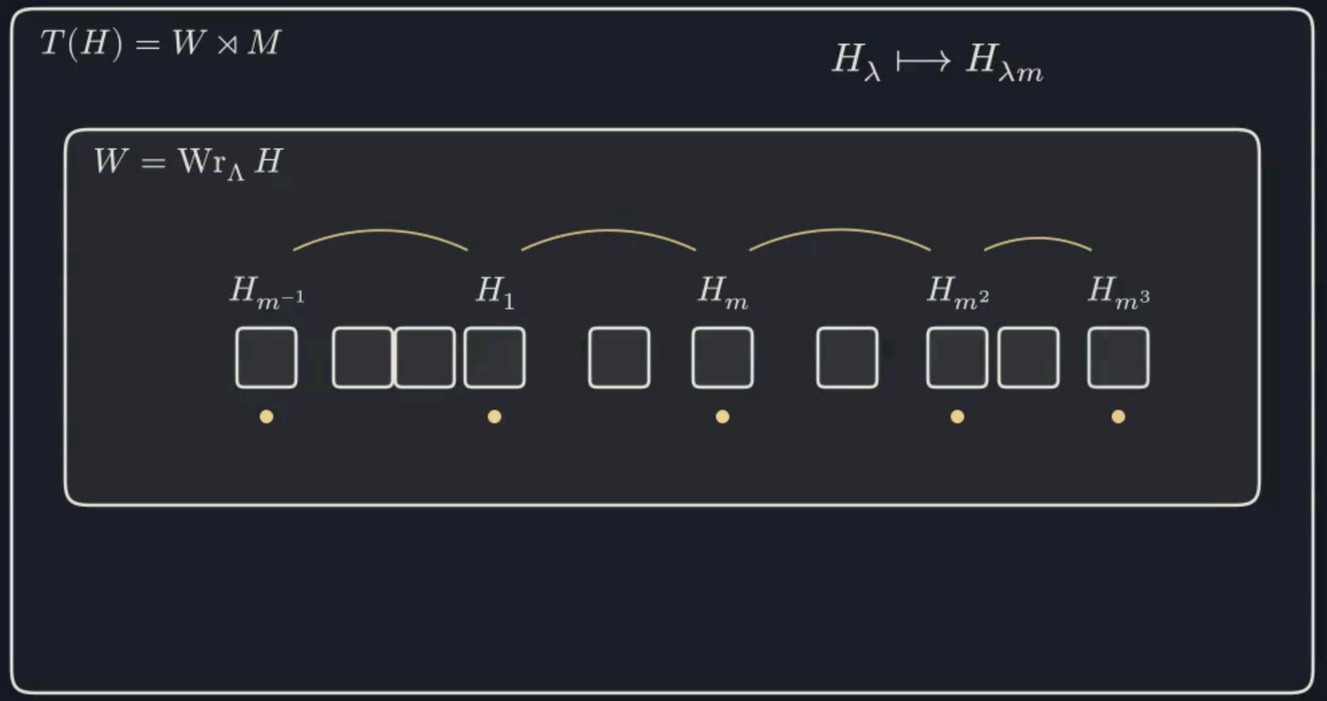
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$$W = \langle H_\lambda : \lambda \in \Lambda \rangle, \qquad H_\lambda = \{h_\lambda : h \in H\} \cong H$$

Since the action of M on Λ is transitive and order-preserving, it induces an action of M on W.

$$T(H) = W \rtimes M$$

All coordinate copies are conjugate in $T(H)$: $m^{-1} h_\lambda m = h_{\lambda m}$, and M is transitive on Λ. So it does not matter which copy is $H \leq W$.



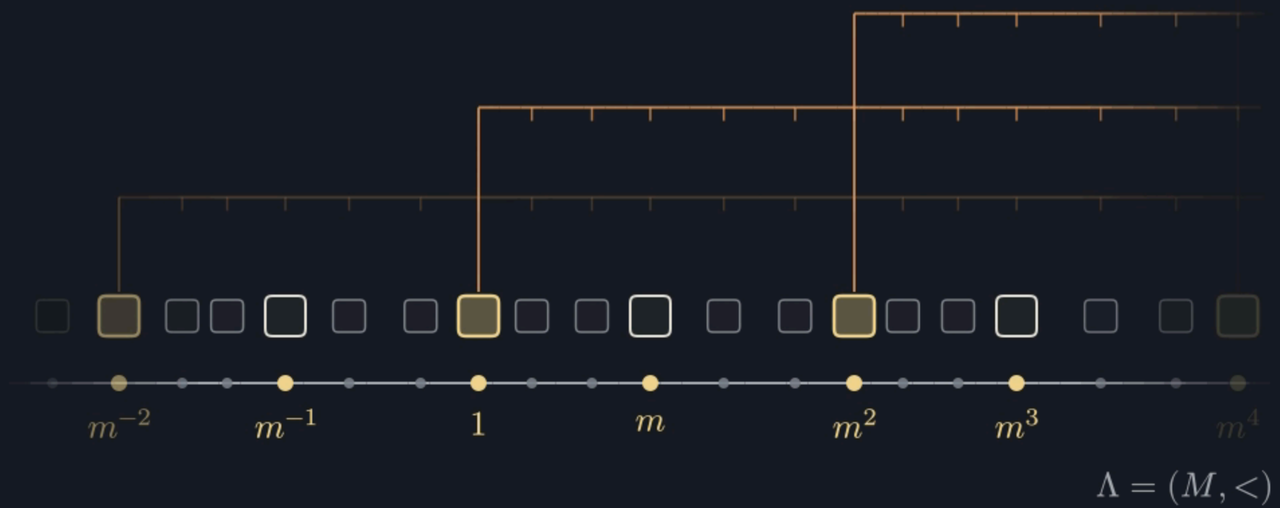
Construction: P. Hall (1962)

One more layer: $T(H)$

$$H_\lambda \longmapsto H_{\lambda m}$$

One $m > 1$ in $M = M(\mathbb{Q}, F)$, applied again and again

$W = \mathrm{Wr}_\Lambda H$



$\Lambda = (M, <)$

$\rho_m : \lambda \longmapsto \lambda m$

$\alpha_m : \ h_\lambda \longmapsto h_{\lambda m} = m^{-1} h_\lambda m$

Copper: h_λ acts only on states that are 1 at every coordinate above λ. Moving h_λ to $h_{\lambda m}$ moves this condition with it.

Gold tiles: the generators $h_{m^{2k}}$, $k \in \mathbb{Z}$. A set of generators of W, not one element of W.

Gold points: the orbit of 1; grey: two other orbits. Positions are an order, not distances, and Λ has many more points.

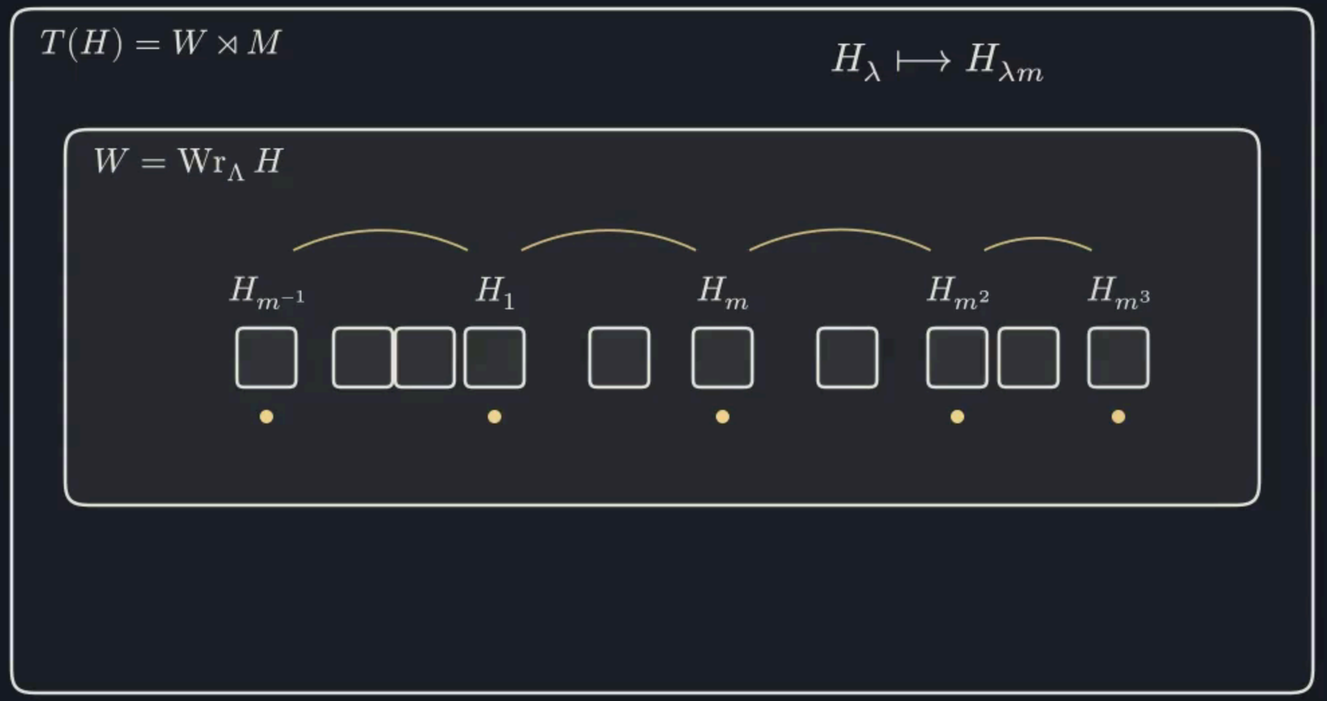
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Construction: P. Hall (1962)

One more layer: $T(H)$

Theorem D of [Hall 1962] implies that W is a minimal normal subgroup of $T(H)$.

Cited, not proved here

$T(H) = W \rtimes M$

$W = \mathrm{Wr}_\Lambda H$



$\langle a \rangle^T = W$

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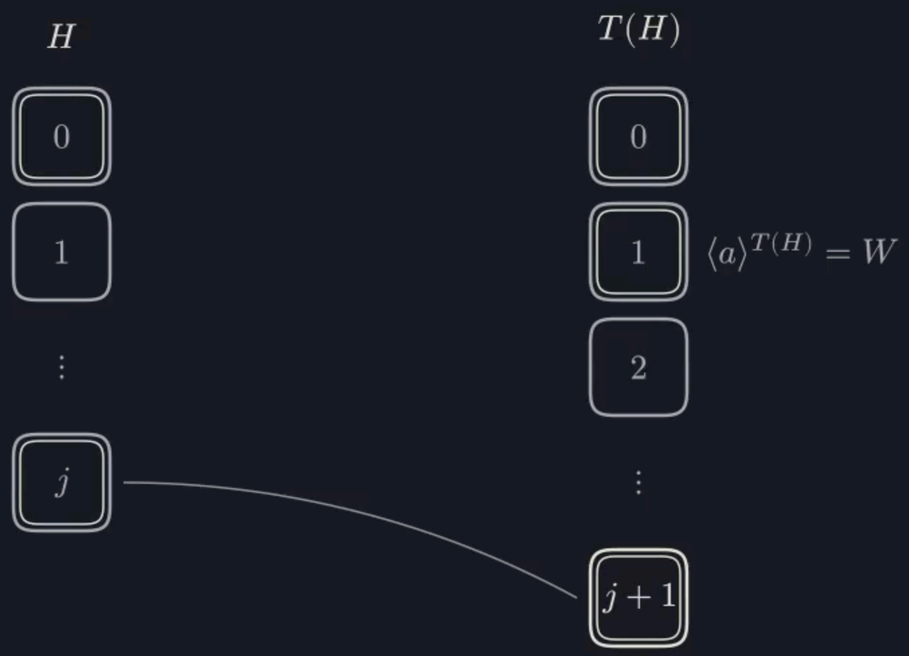
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Hall's hypotheses: $H \neq 1$ is transitive (here regular), and for all $\lambda_1, \lambda_2 \in \Lambda$ some $m \in M$ has $\lambda_2 < \lambda_1 m$. True here: M is transitive and Λ has no maximum. Conclusion: W' is minimal normal, and $W' = W$.

Clearly, since H is perfect, so is W.

By the minimality of W, $\langle a \rangle^T = W$, and therefore $\langle a \rangle^T$ is perfect.



$$\langle a \rangle^{T,j+1} = \langle a \rangle^{W,j}$$

Certified: levels 0, 1 and $j+1$; nothing is claimed about the levels in between or beyond

Theorem 2.8

$\langle a \rangle^{H,j}$ is perfect $\implies$ $\langle a \rangle^{T(H),j+1}$ is perfect

Since $H \leq W$, Lemma 2.4 implies that $\langle a \rangle^{W,j}$ is perfect. $\square$

$$P_0 \leq P_1 \leq \ldots \leq P_N$$

$\langle a \rangle^{P_N, j}$

0

$j > N$

No claim



$$E(H) = \bigoplus_{h \in H} \langle e_h \rangle$$

Each e_h has order 2; H acts by the right regular permutation $e_x^h = e_{xh}$.

$$I(H) = \langle e_1 + e_h : h \in H \rangle \leq E(H)$$

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$$P_{N+1} = T(P_N) \qquad (N \geq 0)$$

Lemma 3.2. Let $N \geq 0$. The group P_N is perfect and has finite normal height. Moreover, there exists an involution $a \in P_0$ such that $\langle a \rangle^{P_N, j}$ is perfect for every $0 \leq j \leq N$.

Just an illustration for $N = 4$

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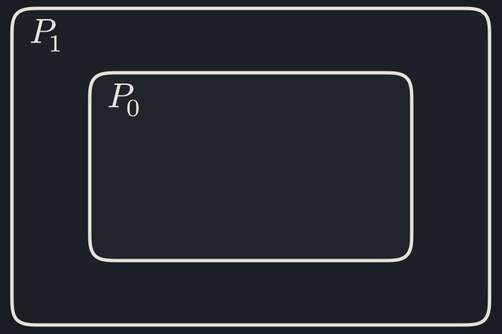
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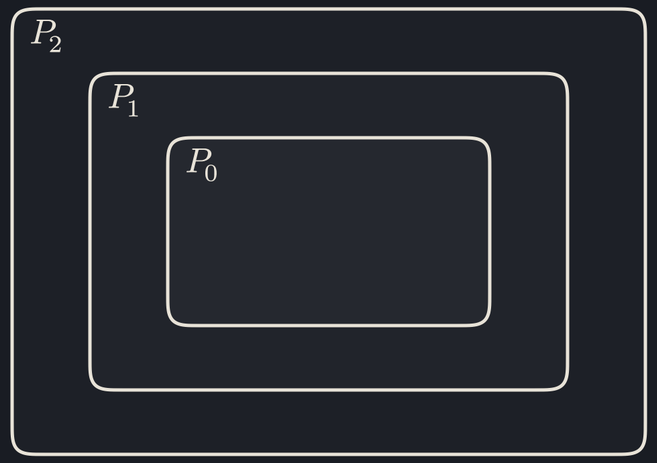
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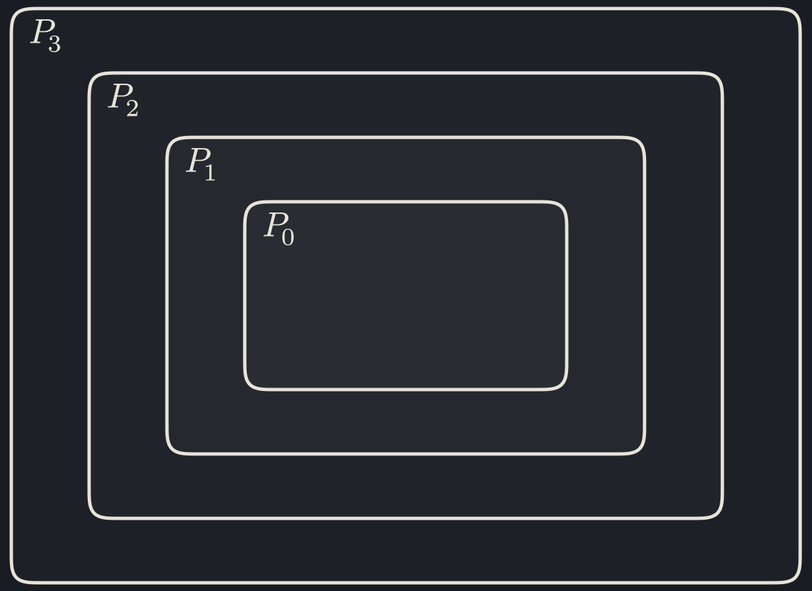
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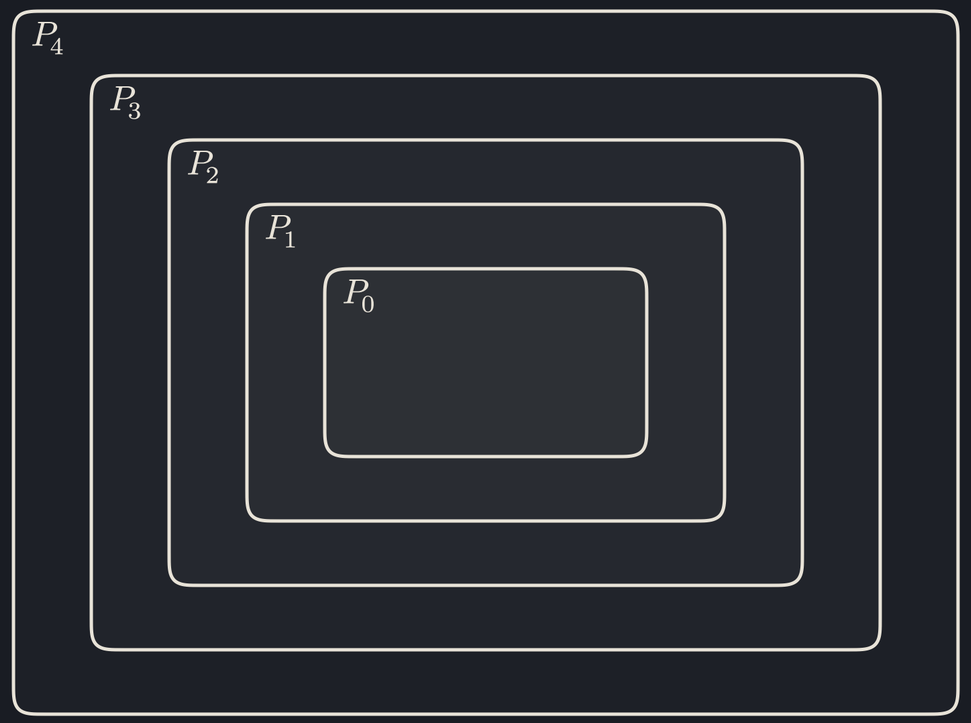
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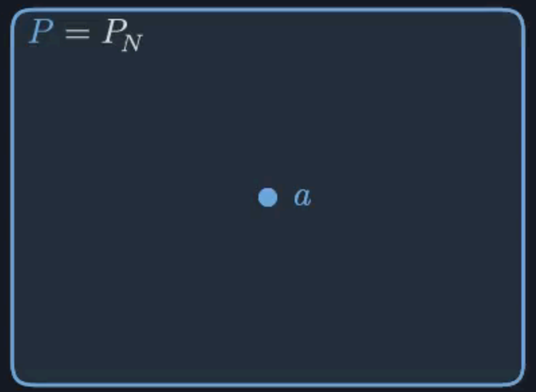
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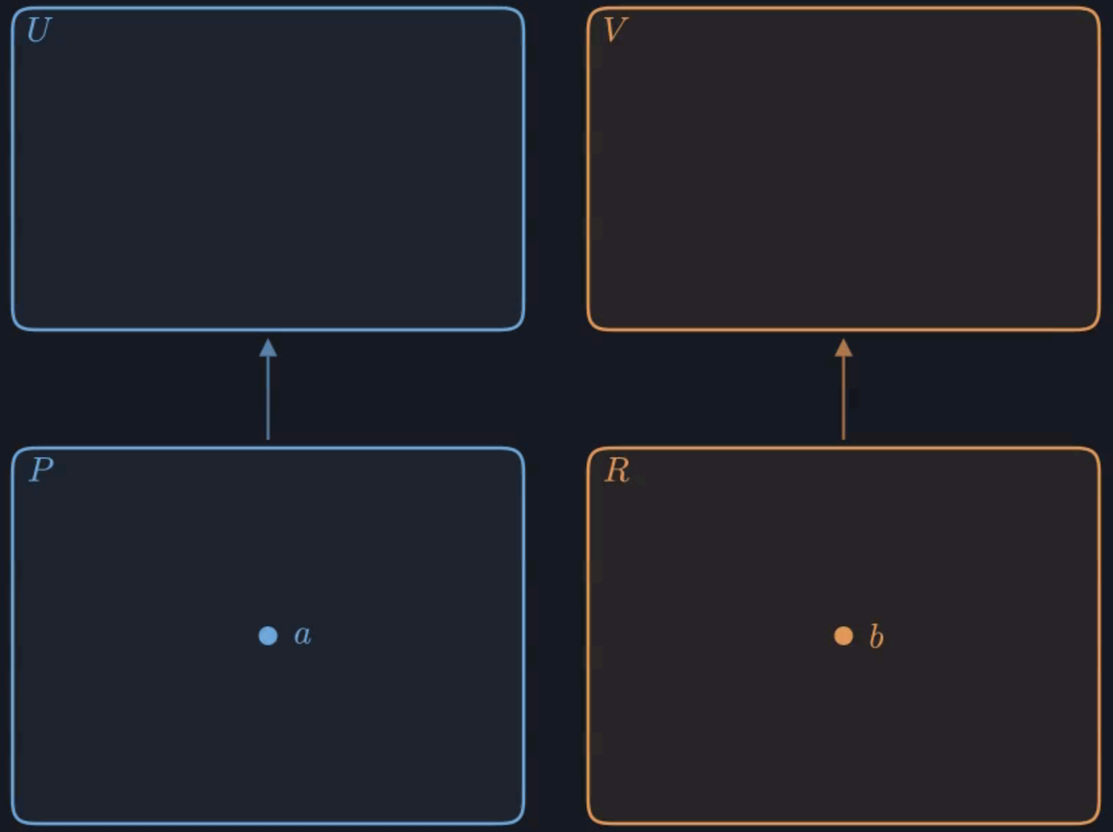


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Let (R, b) be an isomorphic copy of (P, a).



Doubling and twisting: A_N



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Let P act regularly on U, with $u_p^q = u_{pq}$, and trivially on V, and let R act regularly on V and trivially on U.

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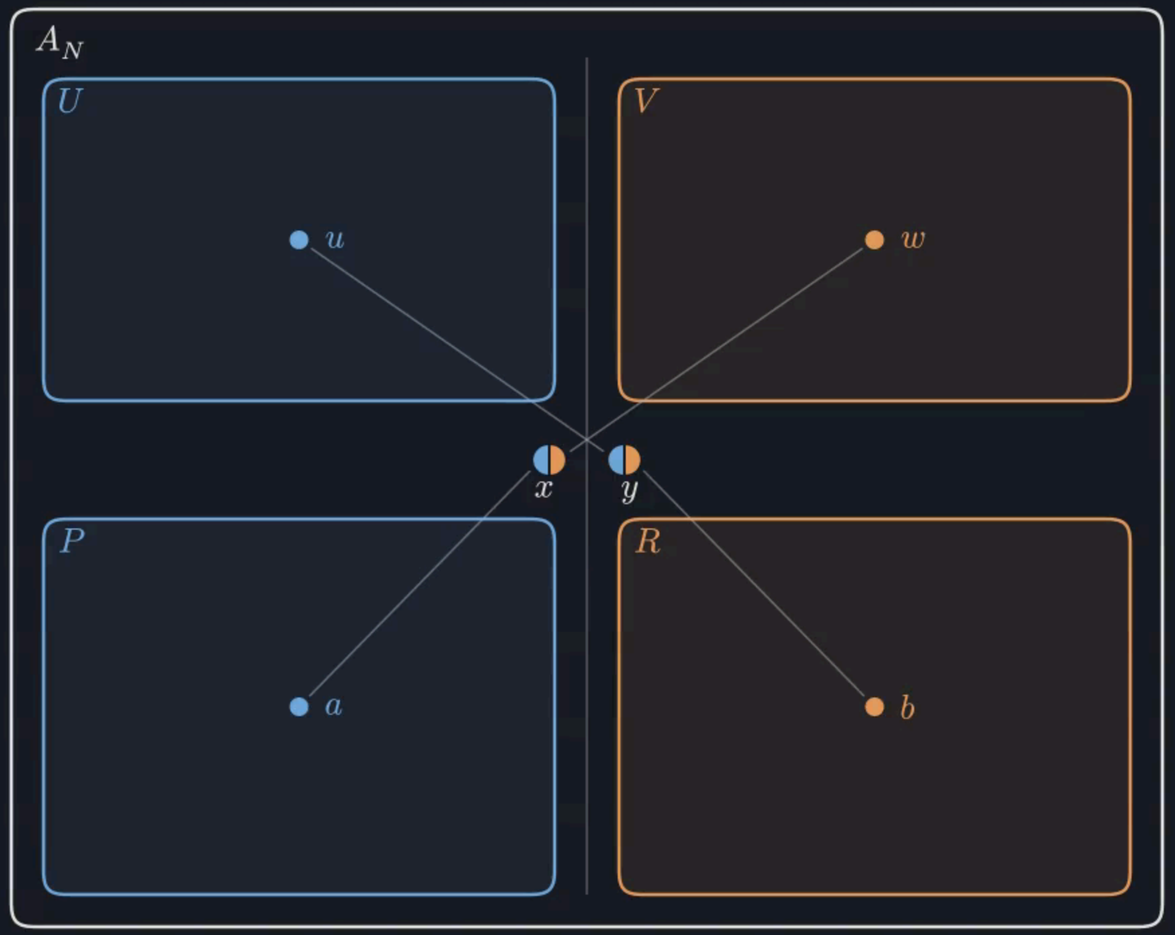
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$$A_N = (U \oplus V) \rtimes (P \times R)$$

Doubling and twisting: A_N



Fix $N \geq 1$, set $P = P_N$, and let $a \in P_0 \leq P$ be the involution given by Lemma 3.2.

Let (R, b) be an isomorphic copy of (P, a).

$$U = \bigoplus_{p \in P} \langle u_p \rangle, \qquad V = \bigoplus_{r \in R} \langle v_r \rangle$$

Let P act regularly on U, with $u_p^q = u_{pq}$, and trivially on V, and let R act regularly on V and trivially on U.

$$A_N = (U \oplus V) \rtimes (P \times R)$$

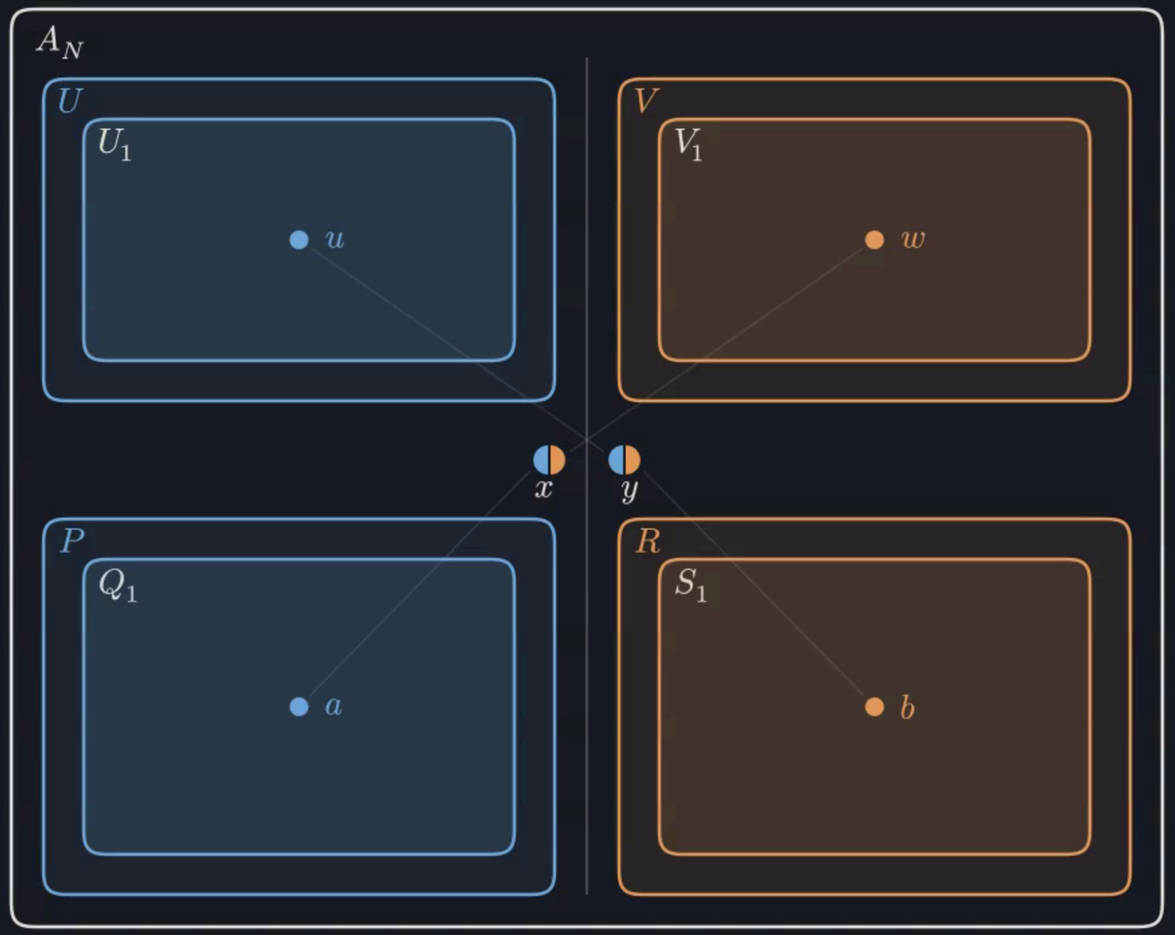
$$u = u_1 + u_a \in U, \qquad w = v_1 + v_b \in V$$

$x = (w, a)$ and $y = (u, b)$

y ◐ ● xy

1 ● ◐ x

$\langle x, y \rangle$ is a Klein 4-group



Proposition 3.3

$$Q_j = \langle a \rangle^{P,j}, \qquad S_j = \langle b \rangle^{R,j}$$

$$U_j = [U, Q_1, \dots, Q_j], \qquad V_j = [V, S_1, \dots, S_j]$$

If $1 \le j \le N$, then $\langle x, y \rangle^{A_N,j} = (U_j \oplus V_j) \rtimes (Q_j \times S_j)$.

In particular, all these N iterated normal closures are perfect.

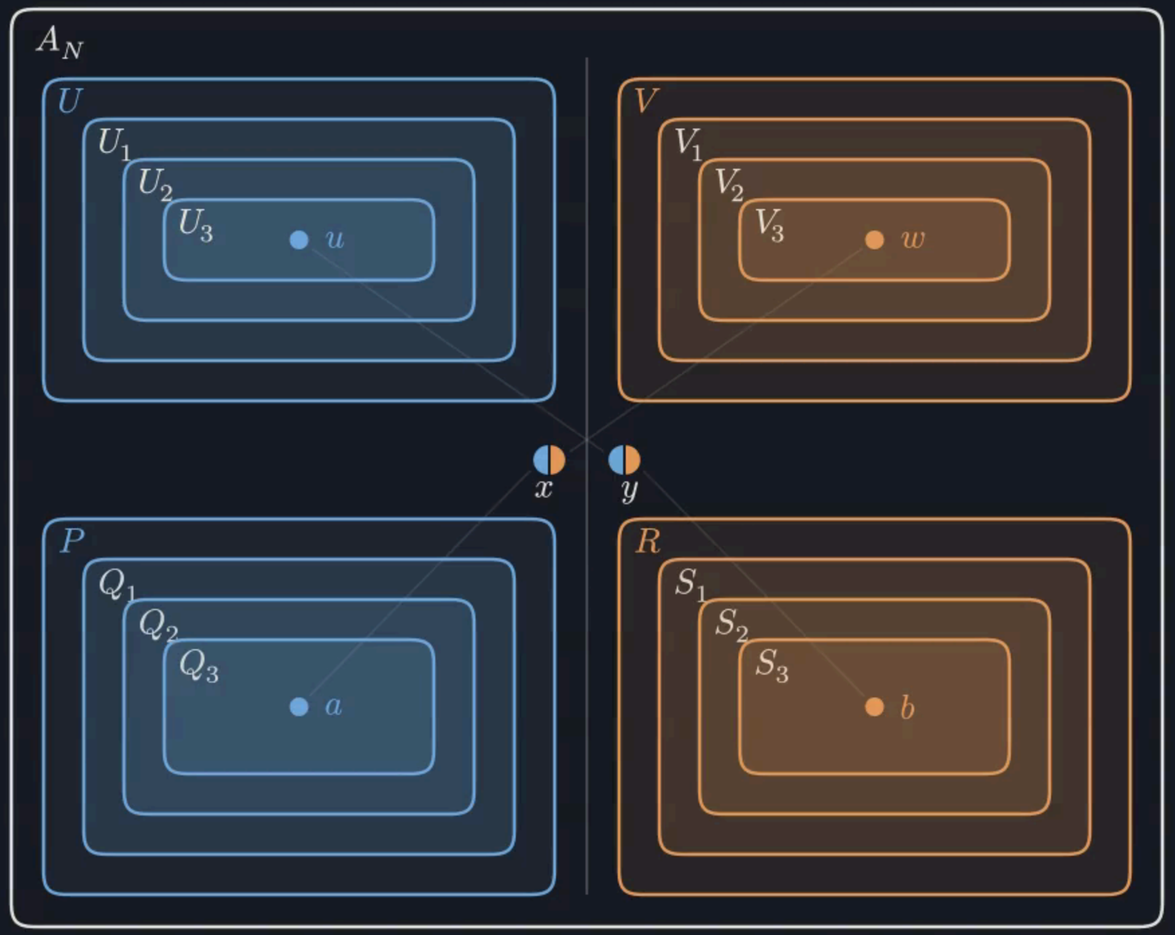


Illustration with $N = 3$

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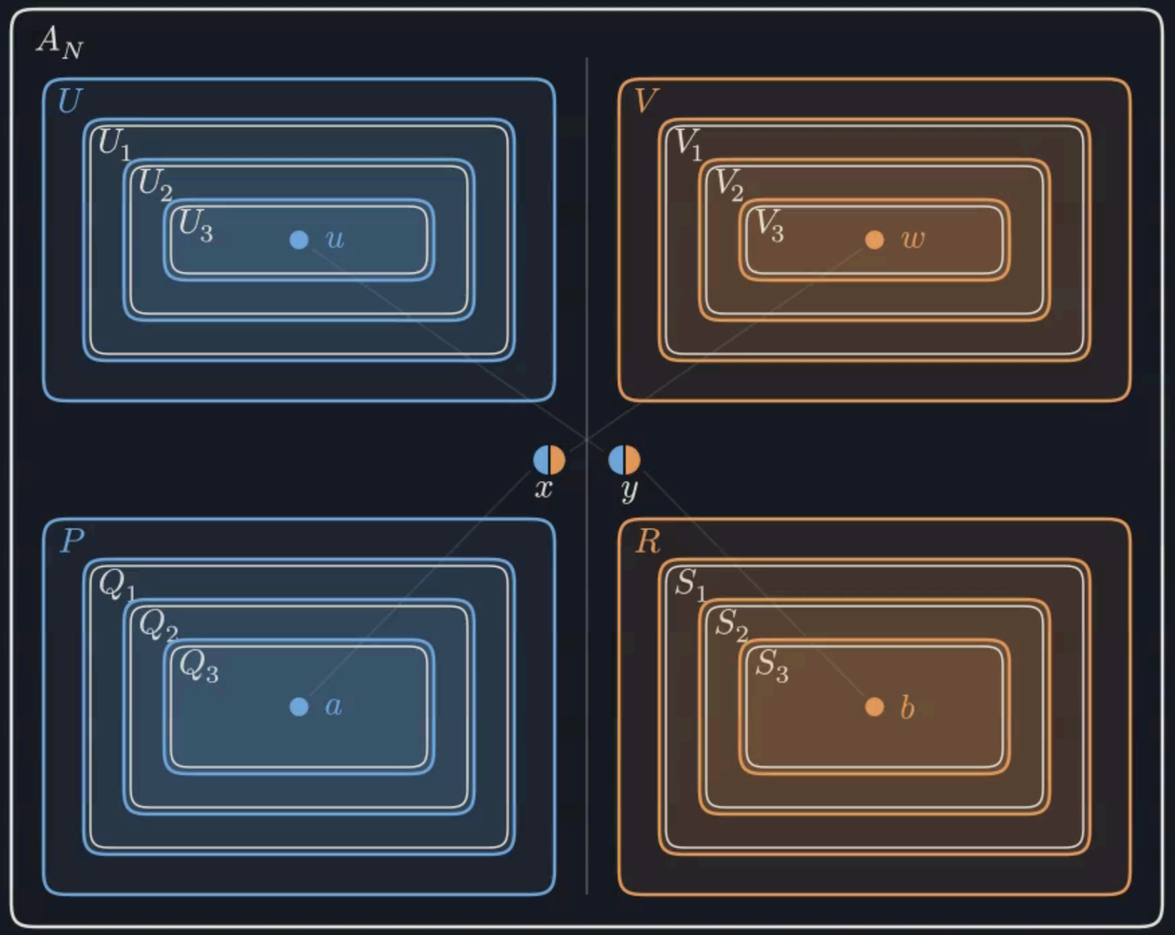


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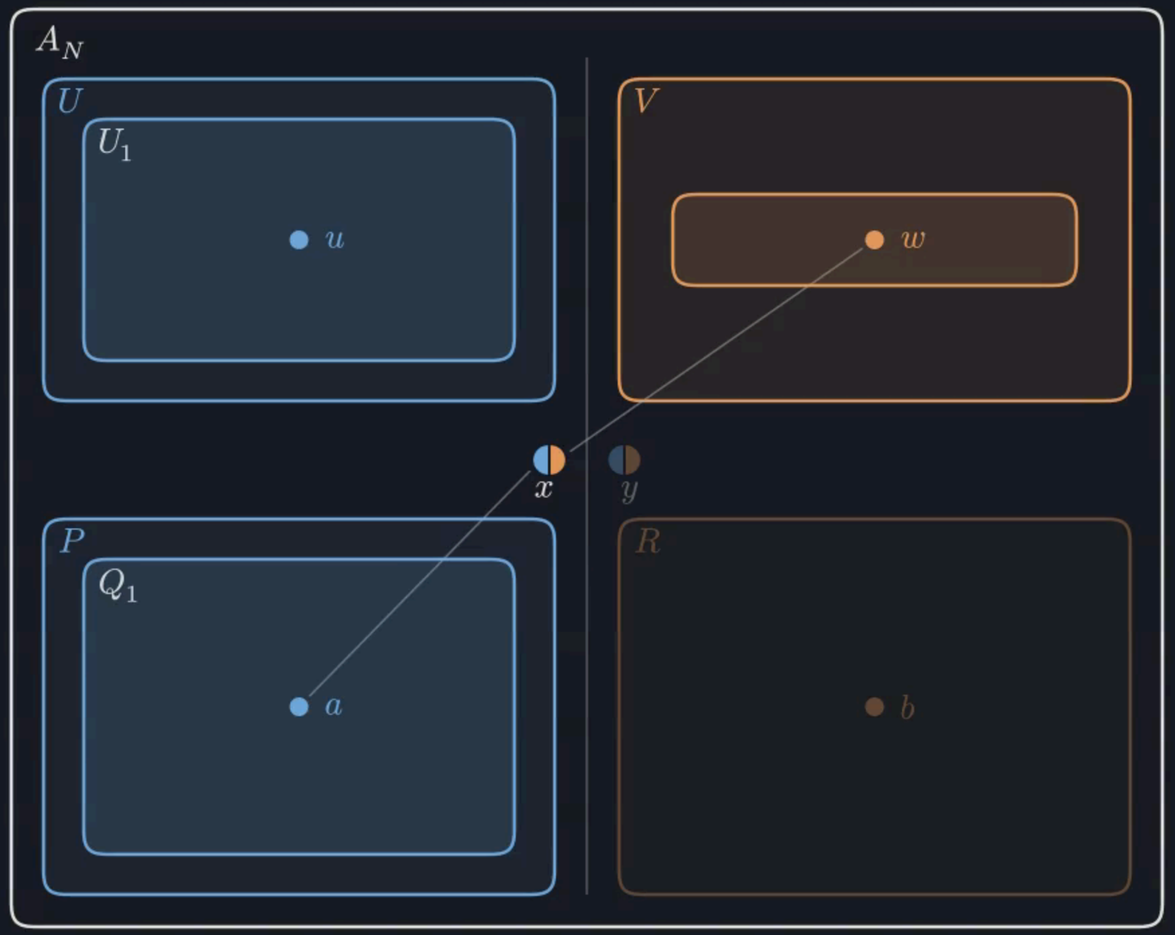
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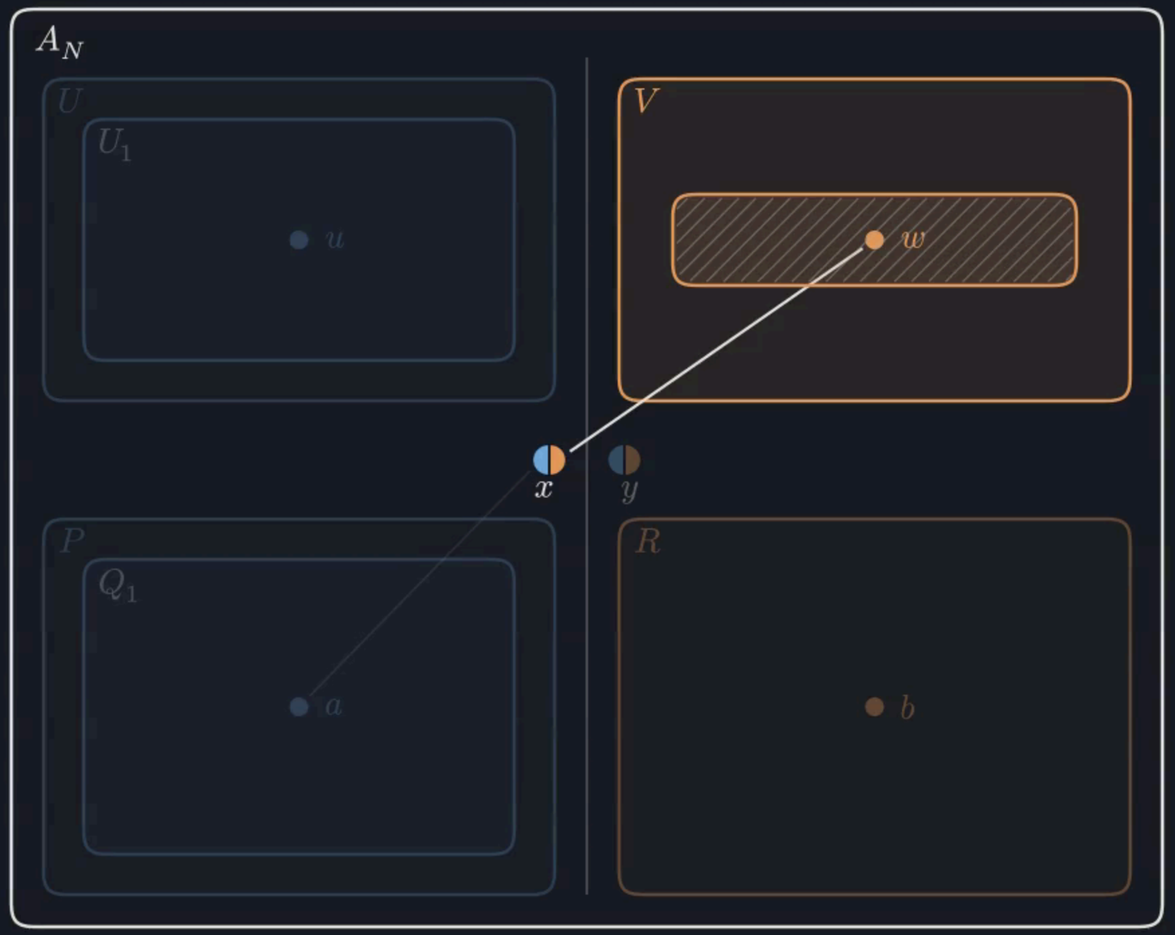
Every Q_j and every S_j is perfect.

Proof. $Q_j \times S_j$ is perfect, and $[U_j, Q_j] = U_j$, $[V_j, S_j] = V_j$ by Lemma 2.2: so $\langle x, y \rangle^{A_N,j}$ is perfect. □



Lemma 3.4

Let $X = \langle x \rangle^{A_N}$. Then X/X' is an elementary abelian 2-group and $x \notin X'$.



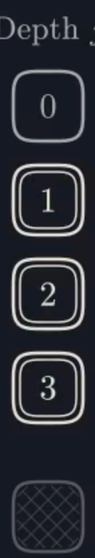
Lemma 3.4

Let $X = \langle x \rangle^{A_N}$. Then X/X' is an elementary abelian 2-group and $x \notin X'$.

Proof. X is generated by conjugates of the involution x, so X/X' is elementary abelian. Modulo the normal subgroup $U \rtimes P$, the group X becomes the abelian group generated by the conjugates of w in V, and x becomes $w = v_1 + v_b \neq 0$. Hence $x \notin X'$. □

The single involution gives non-perfectness

$\langle x, y \rangle^{A_N, j}$

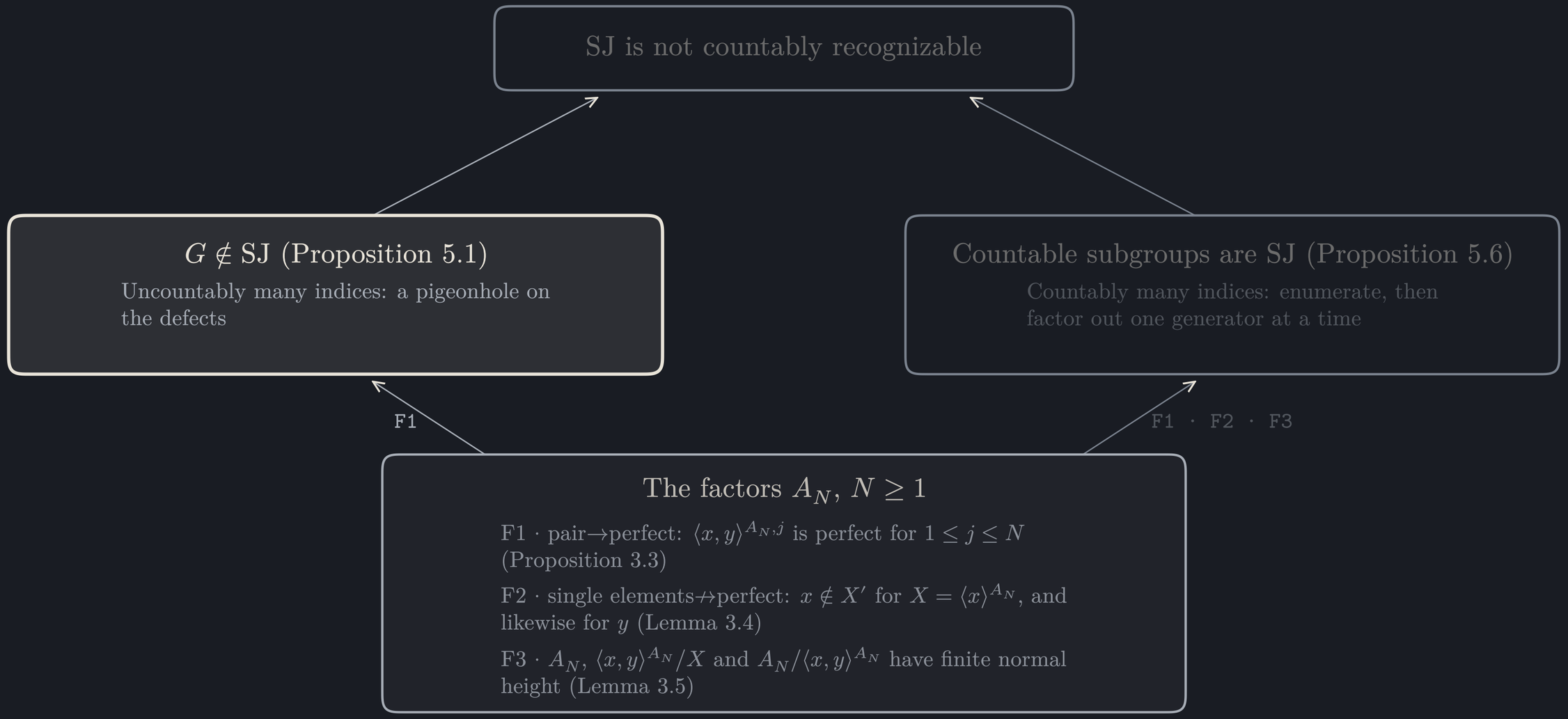


Perfect for $1 \leq j \leq N$

$\langle x \rangle^{A_N}$



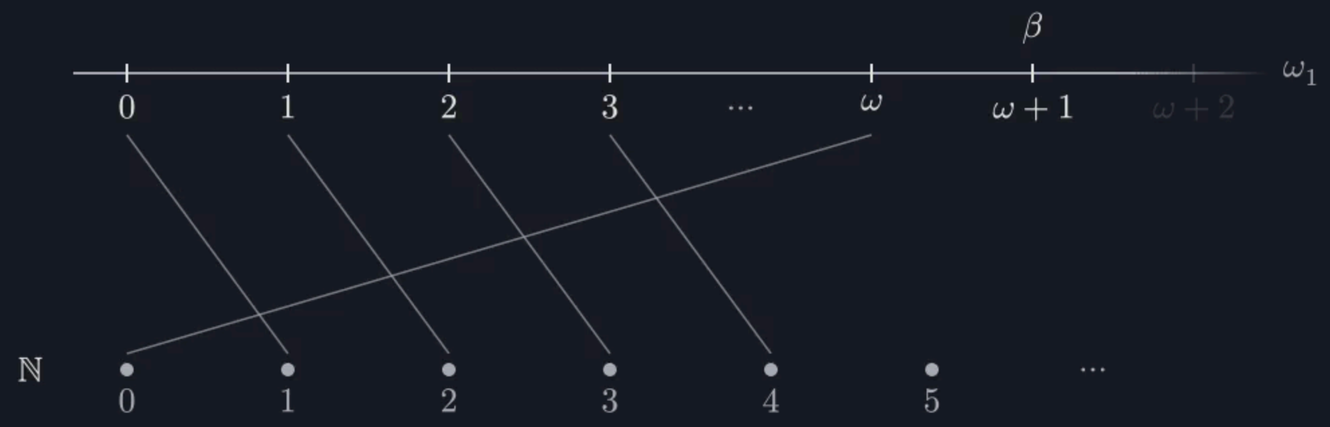
Not perfect: $x \notin X'$


SJ is not countably recognizable
$G \notin$ SJ (Proposition 5.1)
Uncountably many indices: a pigeonhole on the defects
Countable subgroups are SJ (Proposition 5.6)
Countably many indices: enumerate, then factor out one generator at a time
F1
F1 · F2 · F3
The factors A_N, $N \geq 1$
F1 · pair→perfect: $\langle x, y\rangle^{A_N, j}$ is perfect for $1 \leq j \leq N$ (Proposition 3.3)
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F3 · A_N, $\langle x, y\rangle^{A_N}/X$ and $A_N/\langle x, y\rangle^{A_N}$ have finite normal height (Lemma 3.5)

The counterexample

Let ω_1 be the first uncountable ordinal.





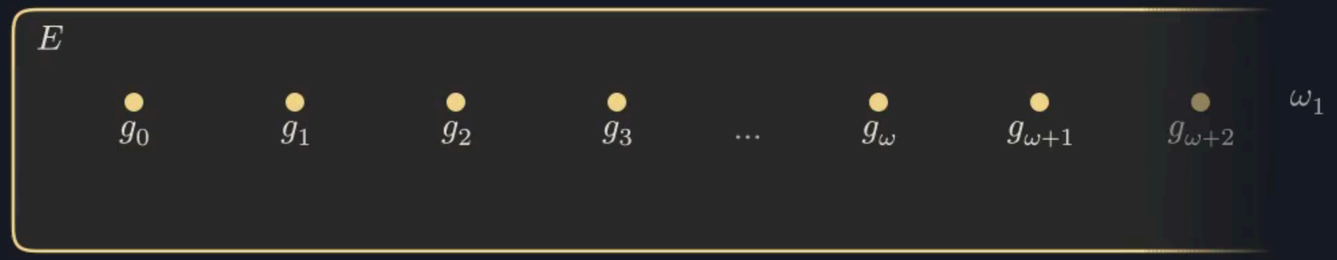
The counterexample

Let ω_1 be the first uncountable ordinal.

For every $\beta < \omega_1$ choose an injection $e_\beta : \beta \longrightarrow \mathbb{N}$, and, for $\alpha < \beta$, put $n(\alpha, \beta) = e_\beta(\alpha) + 1$.

One admissible choice for $\beta = \omega + 1$: $e_{\omega+1}(\omega) = 0$ and $e_{\omega+1}(k) = k + 1$ for $k < \omega$.

$$n(\omega, \omega+1) = 1, \quad n(0, \omega+1) = 2,$$
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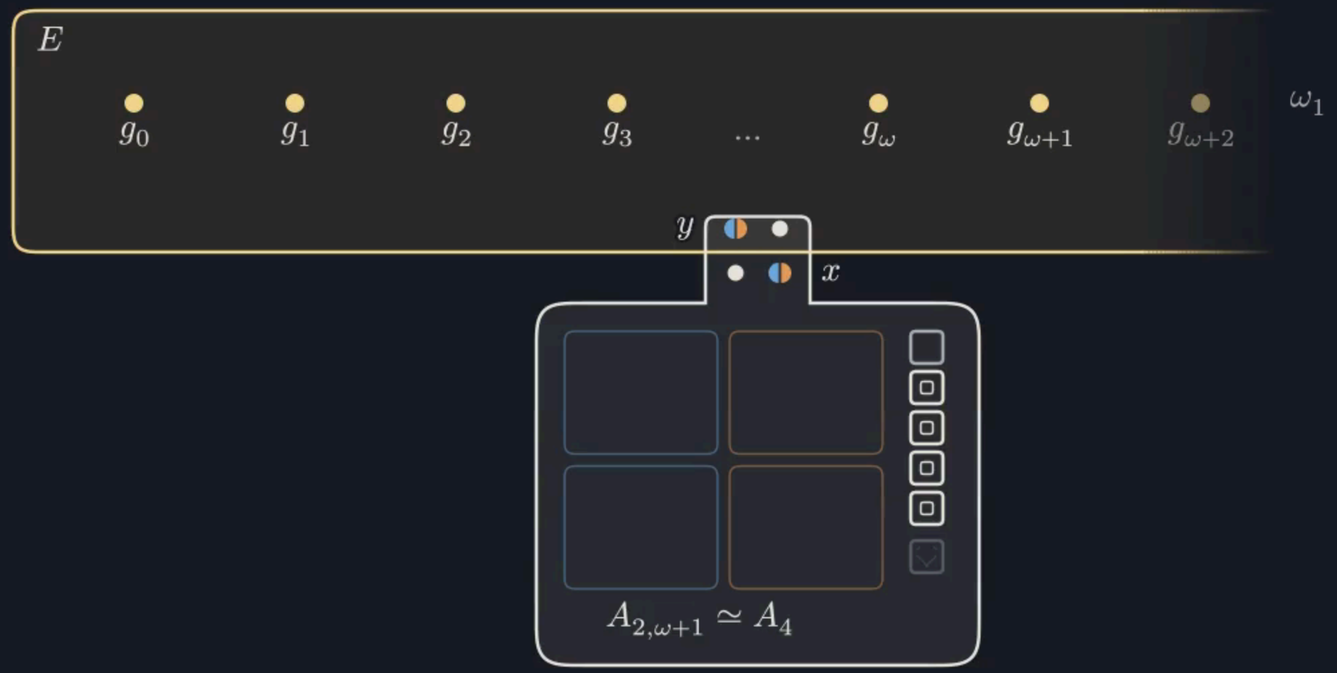
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$$E = \bigoplus_{\alpha<\omega_1} \langle g_\alpha \rangle, \qquad |g_\alpha| = 2$$



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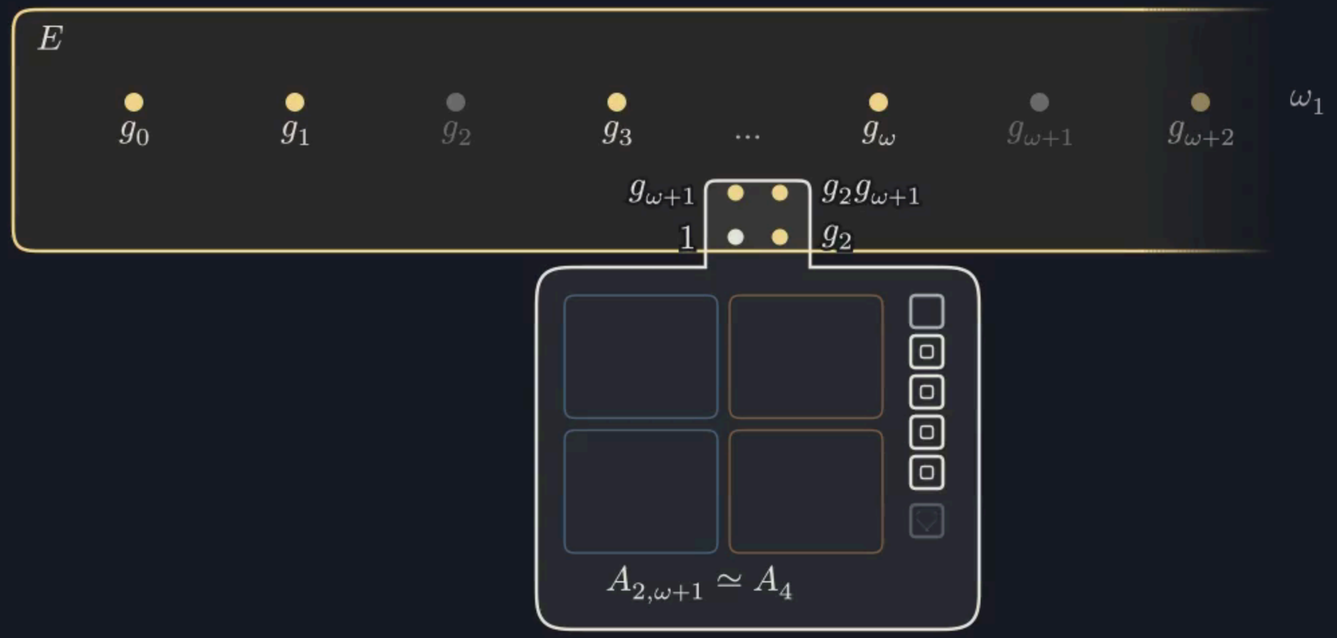
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$$E = \bigoplus_{\alpha<\omega_1} \langle g_\alpha \rangle, \qquad |g_\alpha| = 2$$

For each $\alpha < \beta$ take a copy $A_{\alpha\beta}$ of the group $A_{n(\alpha,\beta)}$ constructed above, with commuting involutions $x_{\alpha\beta}, y_{\alpha\beta}$

$$A_{2,\omega+1} \simeq A_{n(2,\omega+1)} = A_4$$



The counterexample

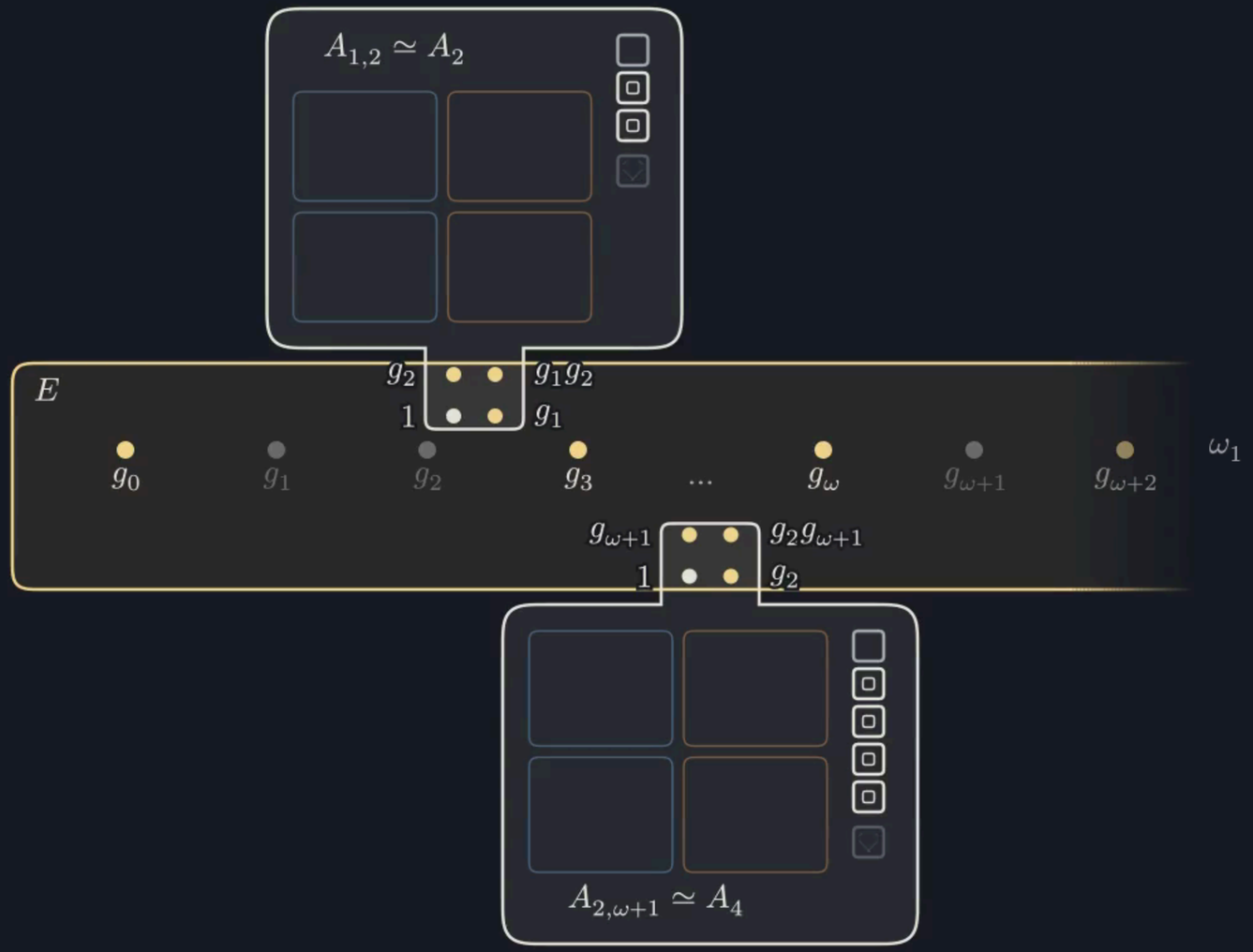
Define G by the presentation obtained from

$$E * \underset{\alpha<\beta<\omega_1}{*} A_{\alpha\beta}$$

by imposing the identifications $x_{\alpha\beta} \mapsto g_\alpha$ and $y_{\alpha\beta} \mapsto g_\beta$.

$$x_{2,\omega+1} \mapsto g_2, \qquad y_{2,\omega+1} \mapsto g_{\omega+1}$$

For each pair $\alpha < \beta$, these two relations identify the Klein four group $\langle x_{\alpha\beta}, y_{\alpha\beta} \rangle$ with the Klein four subgroup $\langle g_\alpha, g_\beta \rangle$ of E.

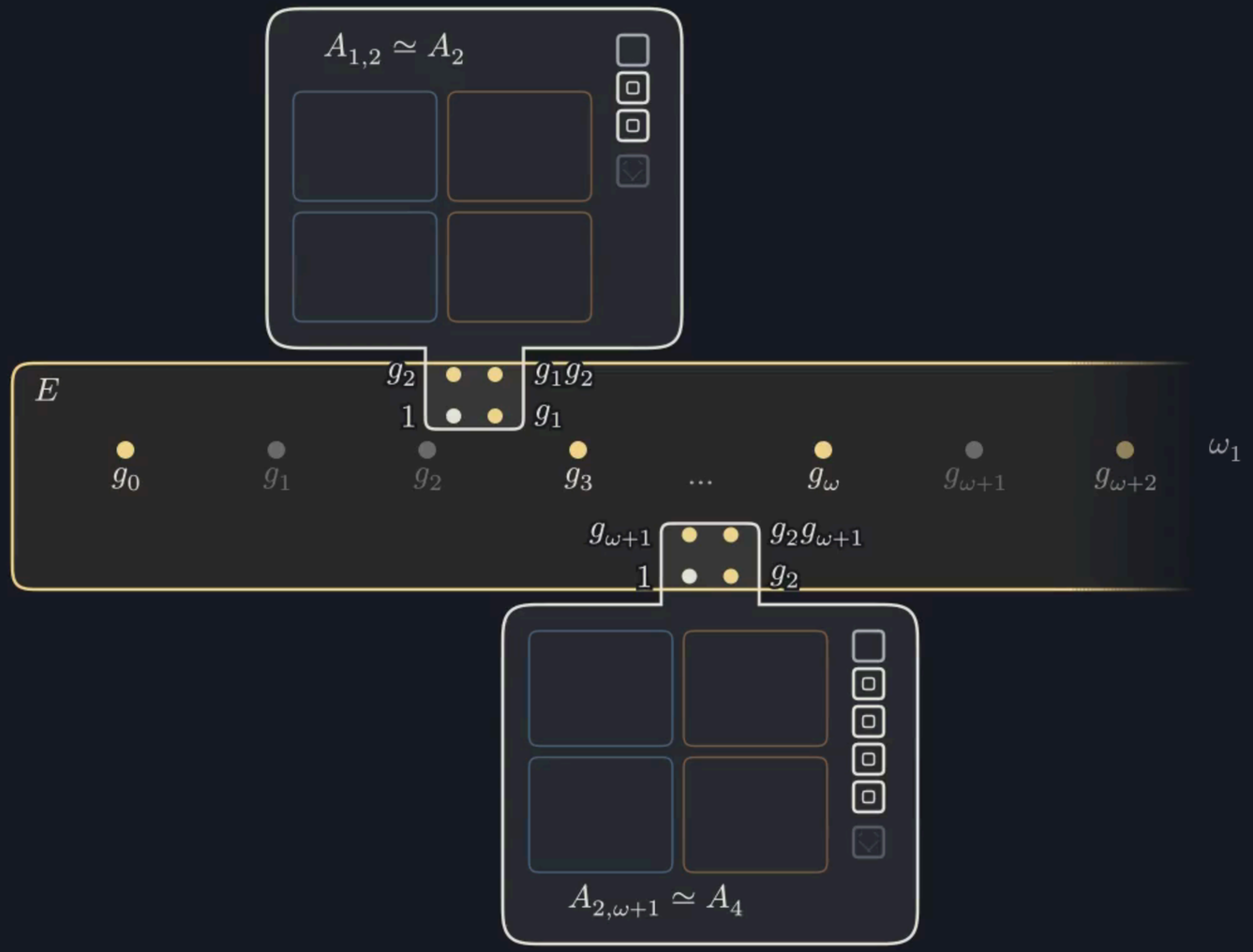


The counterexample

$A_{1,2} \simeq A_{n(1,2)} = A_2$

$x_{1,2} \mapsto g_1, \qquad y_{1,2} \mapsto g_2$

g_2 is identified with $x_{2,\omega+1}$ and with $y_{1,2}$.

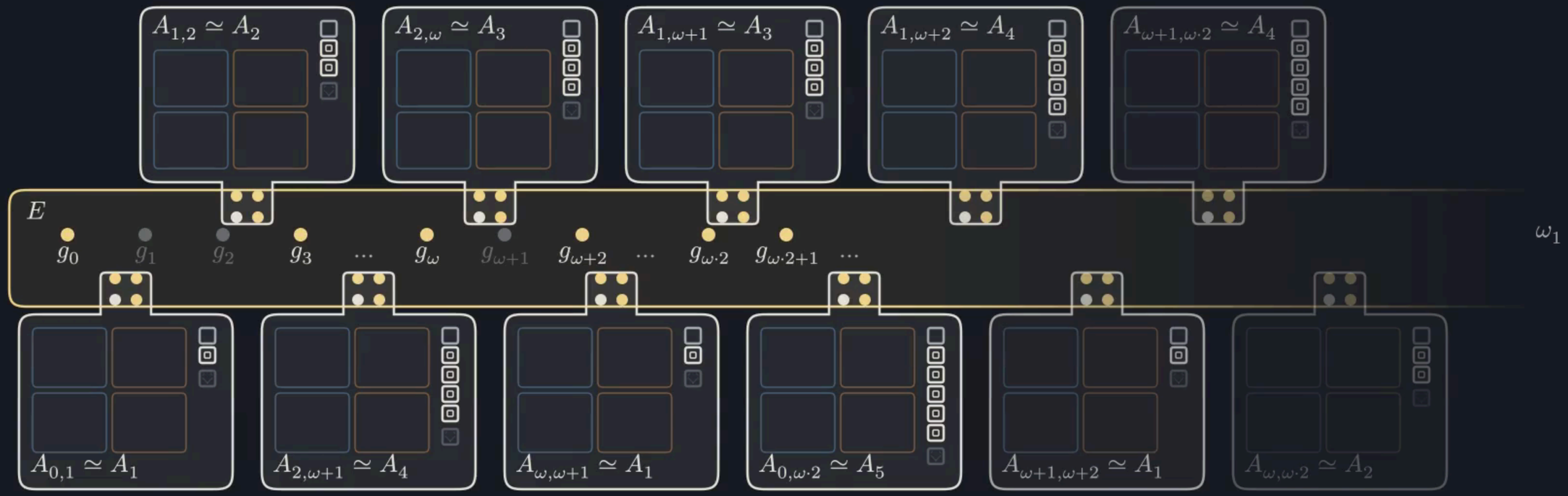


The counterexample

After well-ordering the set of pairs $\alpha < \beta < \omega_1$, we may therefore construct G by adjoining the factors $A_{\alpha\beta}$ one at a time to the initial factor E over these Klein four subgroups, and by taking unions at limit stages.

Thus G is an iterated amalgam in the sense of Section 4.


$A_{1,2} \simeq A_2$
E
g_0
g_1
g_2
g_3
$\cdots$
g_ω
$g_{\omega+1}$
$g_{\omega+2}$
$\cdots$
$g_{\omega\cdot 2}$
$g_{\omega\cdot 2+1}$
$\cdots$
ω_1
$A_{2,\omega+1} \simeq A_4$

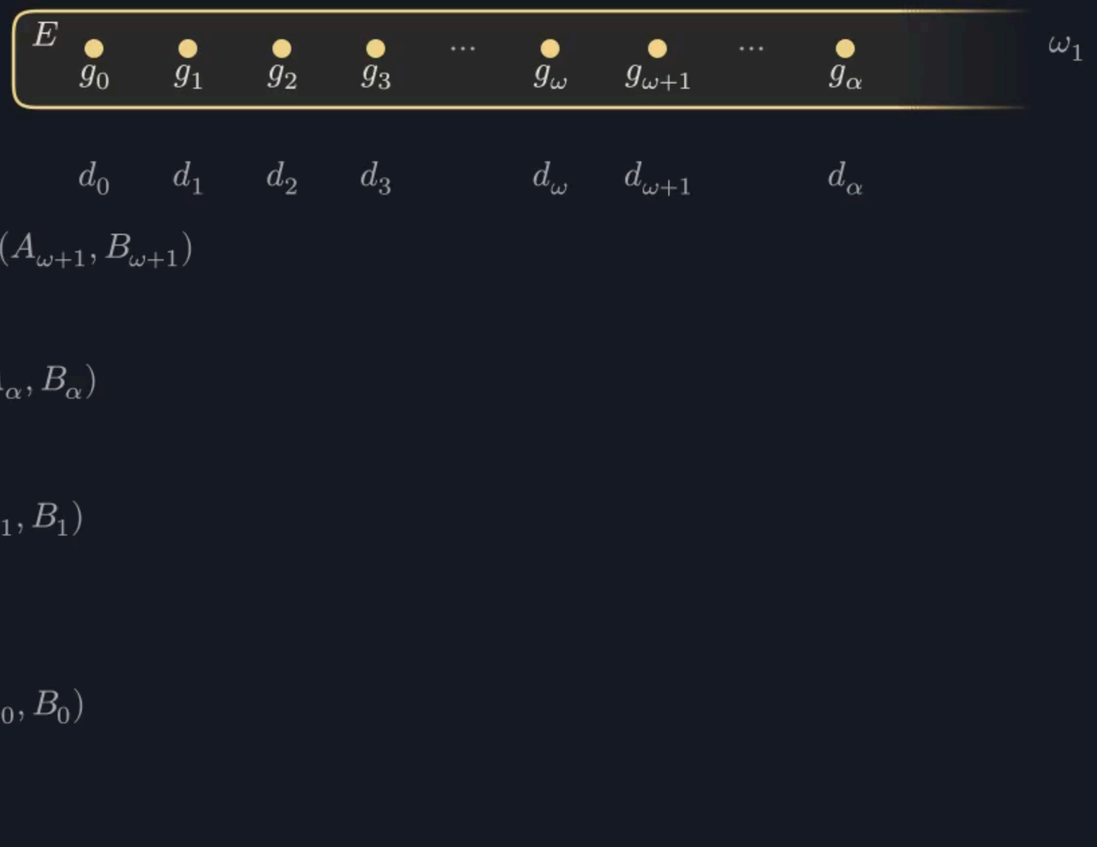


A finite sample of the factors $A_{\alpha\beta}$, $\alpha < \beta < \omega_1$. Each one meets E only in its Klein four group.

The normal-form theorem at the successor stages, together with the unions at limit stages, shows that E and every $A_{\alpha\beta}$ embed in G.

G is not SJ

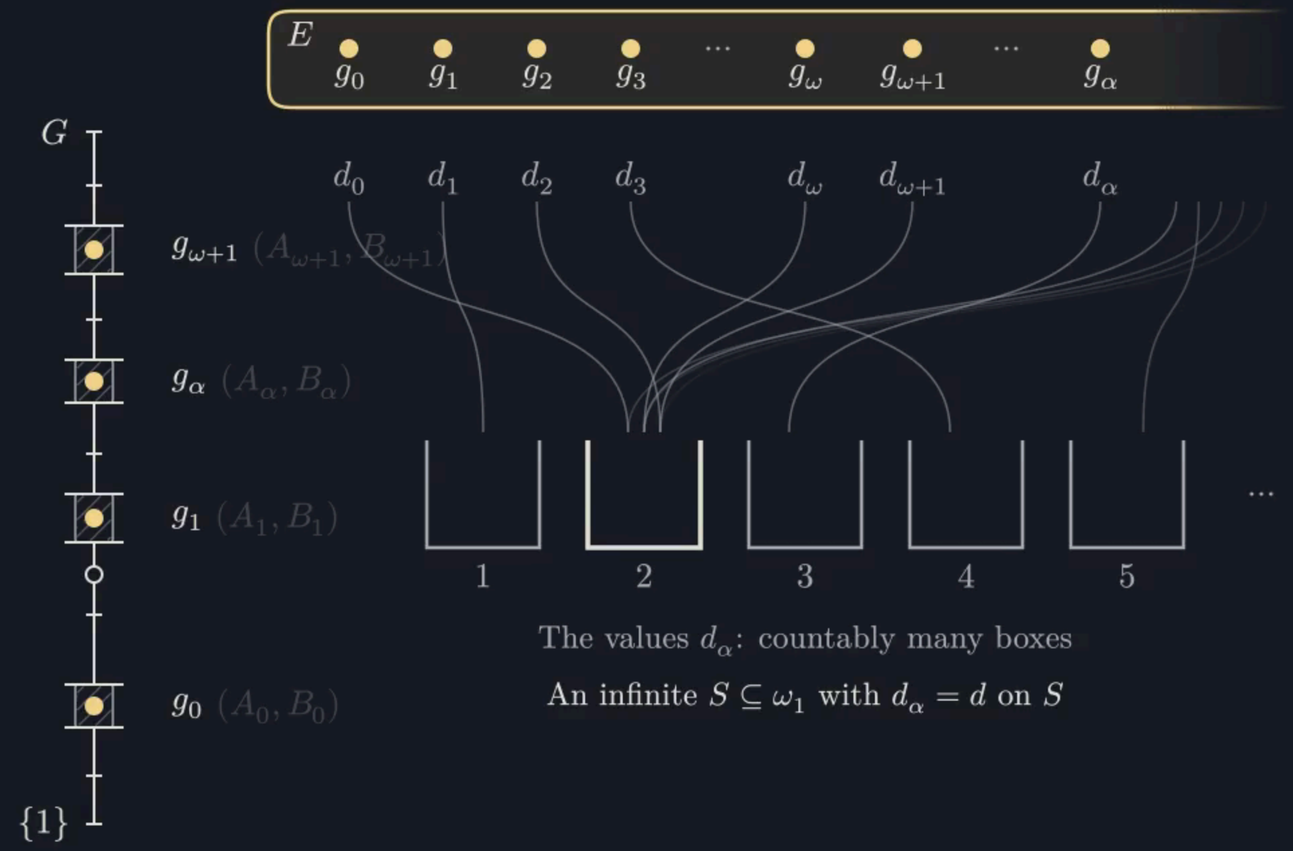
Suppose that G has an SJ-series $\mathcal{S}$.



Proposition 5.1. The group G does not belong to SJ.

Proof. $d_\alpha = \max\{\mathrm{d}_G(A_\alpha), \mathrm{d}_G(B_\alpha)\}$

Suppose that G has an SJ-series $\mathcal{S}$.



The values d_α: countably many boxes

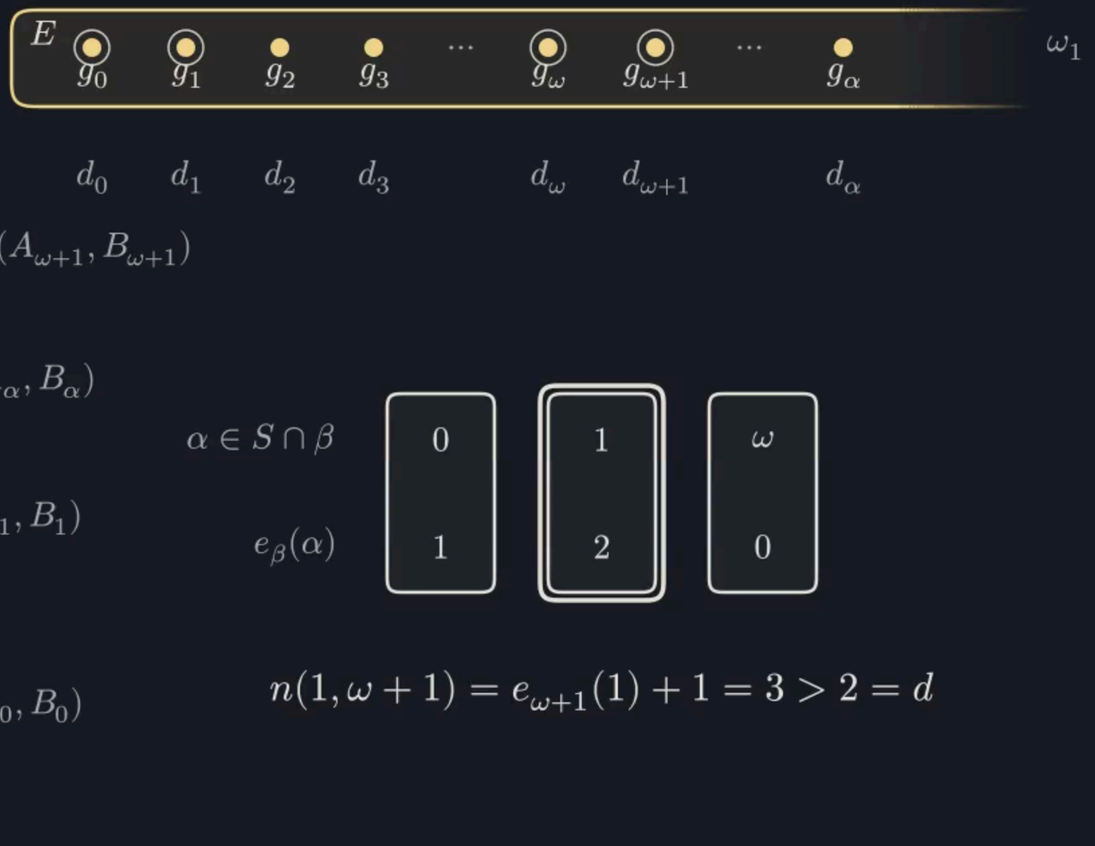
An infinite $S \subseteq \omega_1$ with $d_\alpha = d$ on S

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$d_\alpha = d$ for $\alpha \in S$, $S \subseteq \omega_1$ infinite

Suppose that G has an SJ-series $\mathcal{S}$.



$$n(1, \omega+1) = e_{\omega+1}(1) + 1 = 3 > 2 = d$$

Illustration: $d = 2$, $\beta = \omega + 1$ and $S \cap \beta \supseteq \{0, 1, \omega\}$, with the admissible $e_{\omega+1}$ chosen before

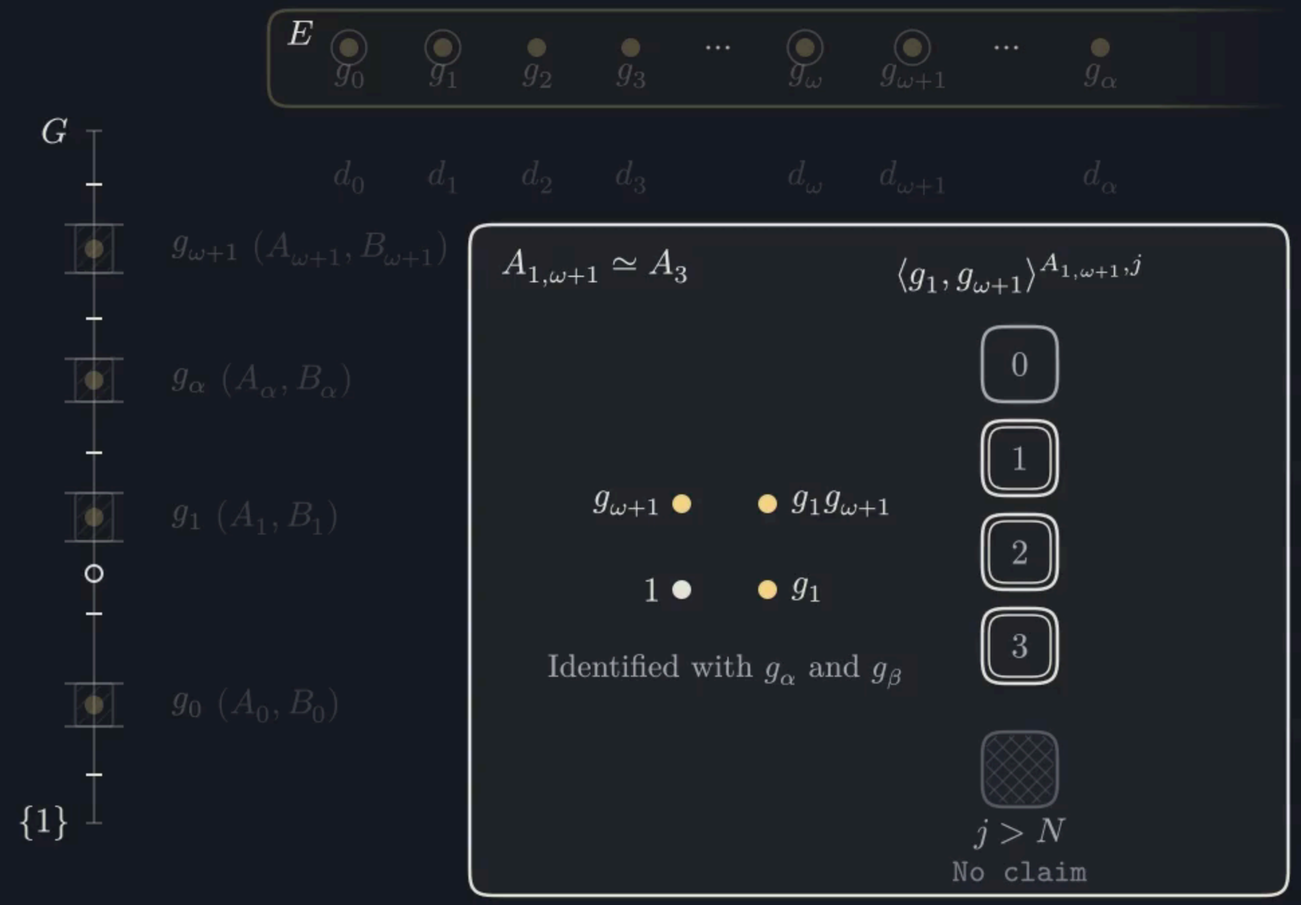
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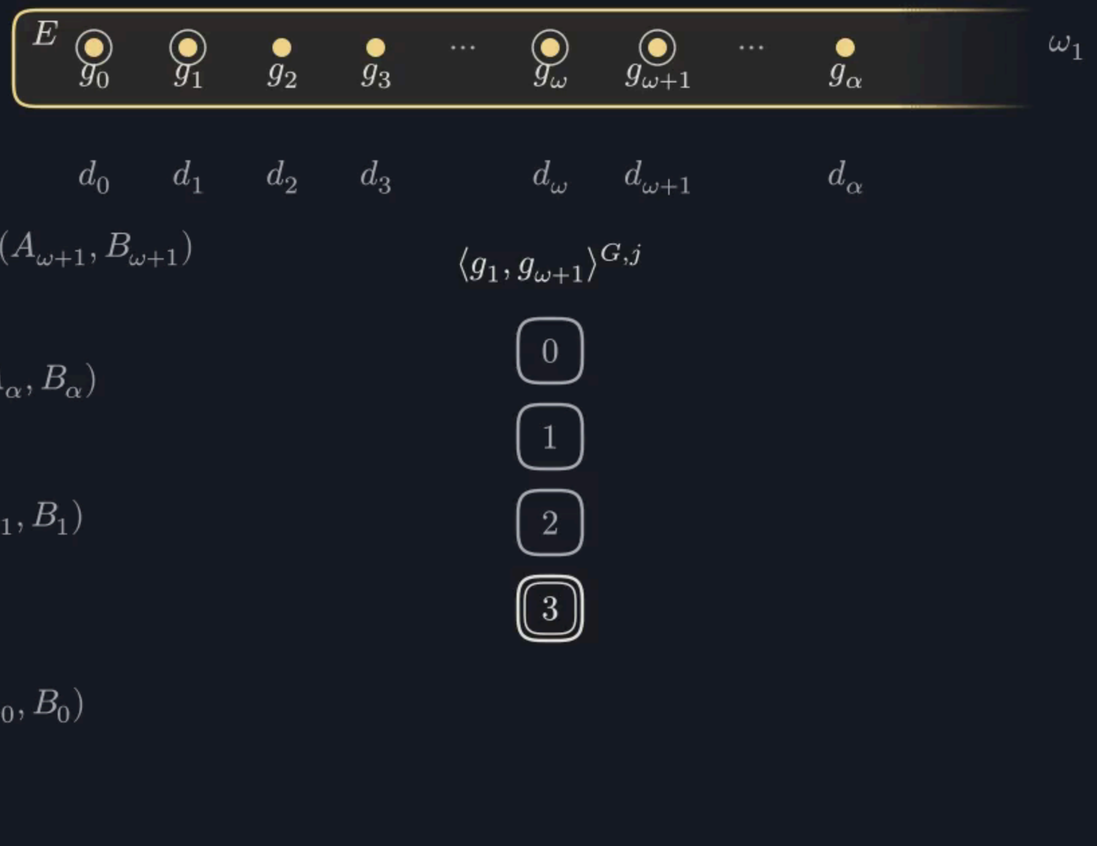
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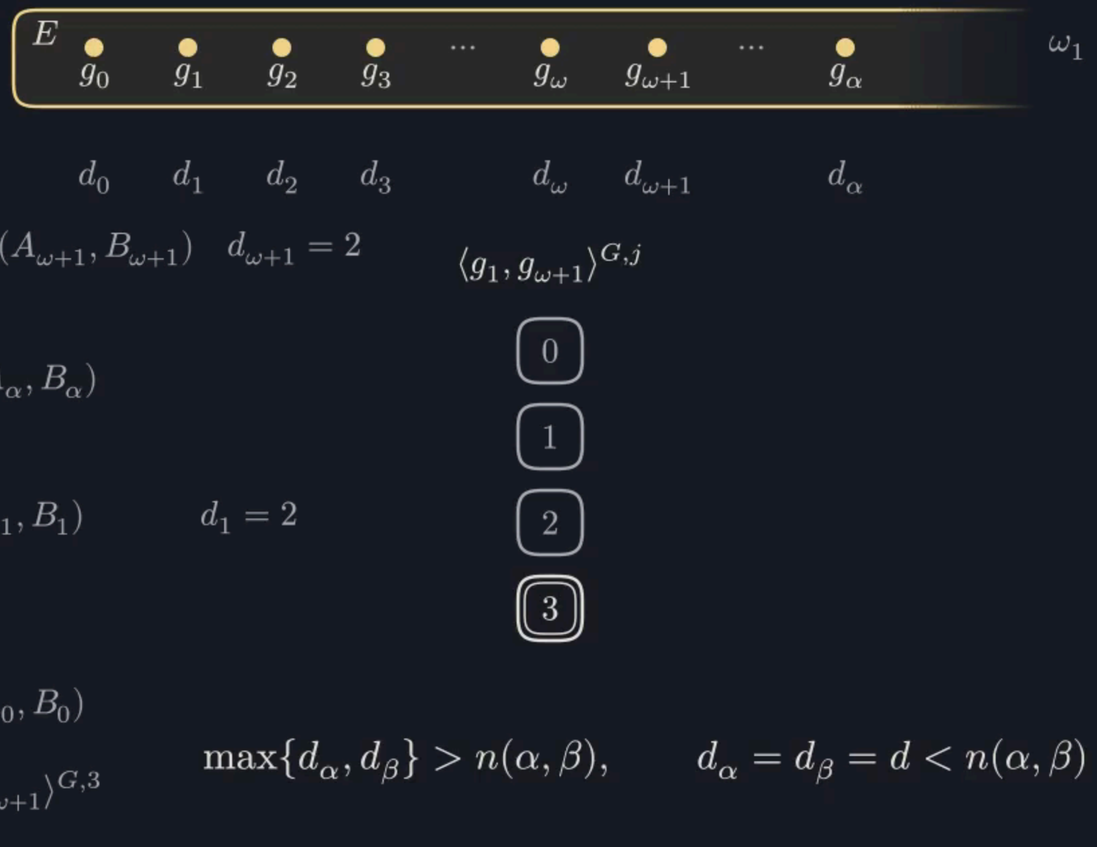
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$\langle g_\alpha, g_\beta\rangle^{G, n(\alpha,\beta)}$ is perfect (Lemma 2.4)

~~Suppose that G has an SJ series $\mathcal{S}$.~~



$$\max\{d_\alpha, d_\beta\} > n(\alpha,\beta), \qquad d_\alpha = d_\beta = d < n(\alpha,\beta)$$

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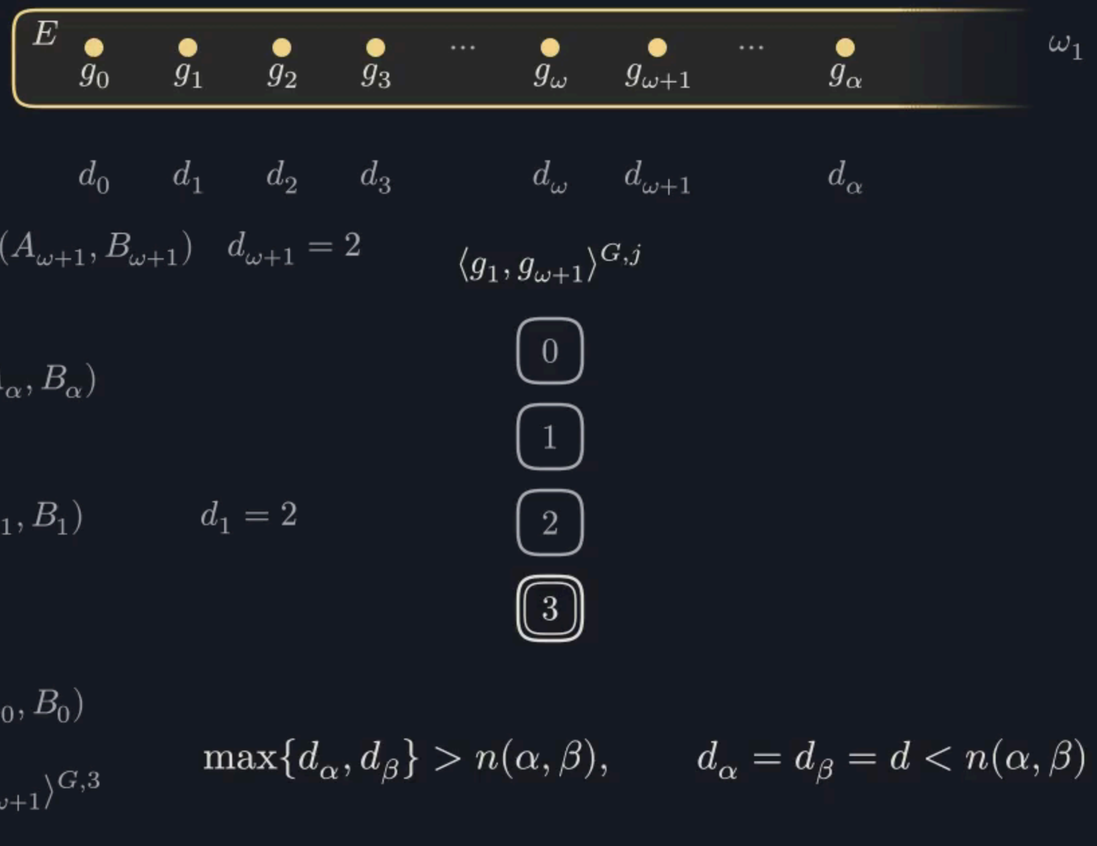
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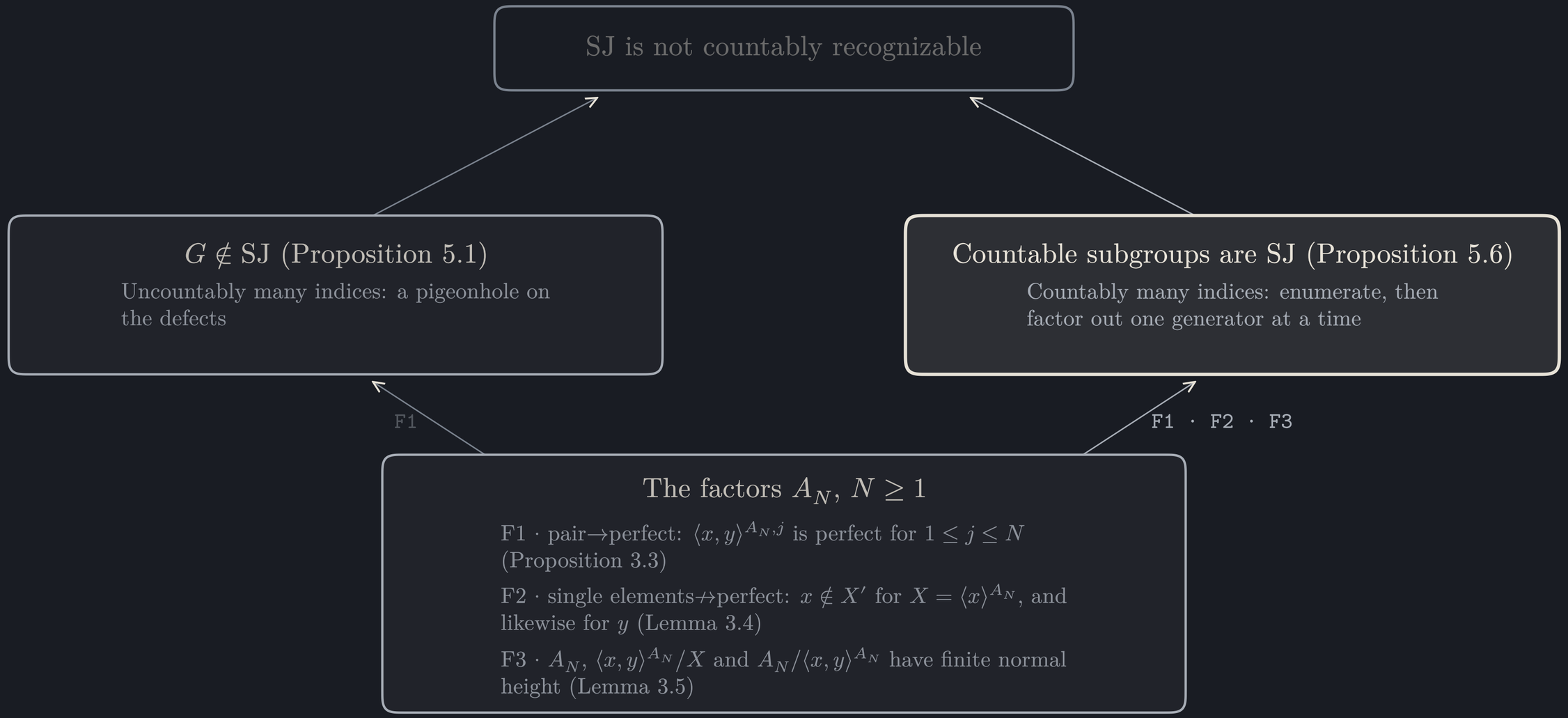
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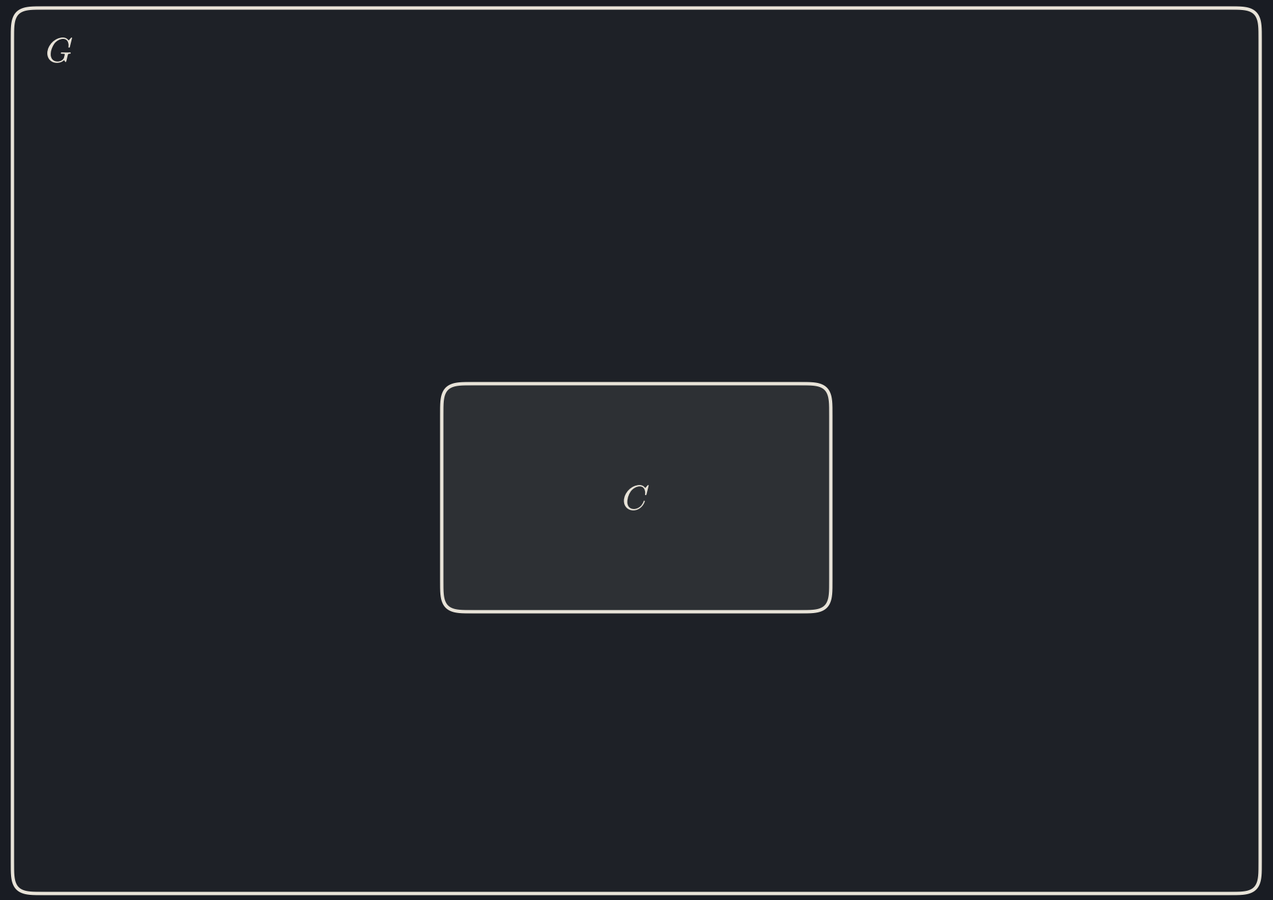
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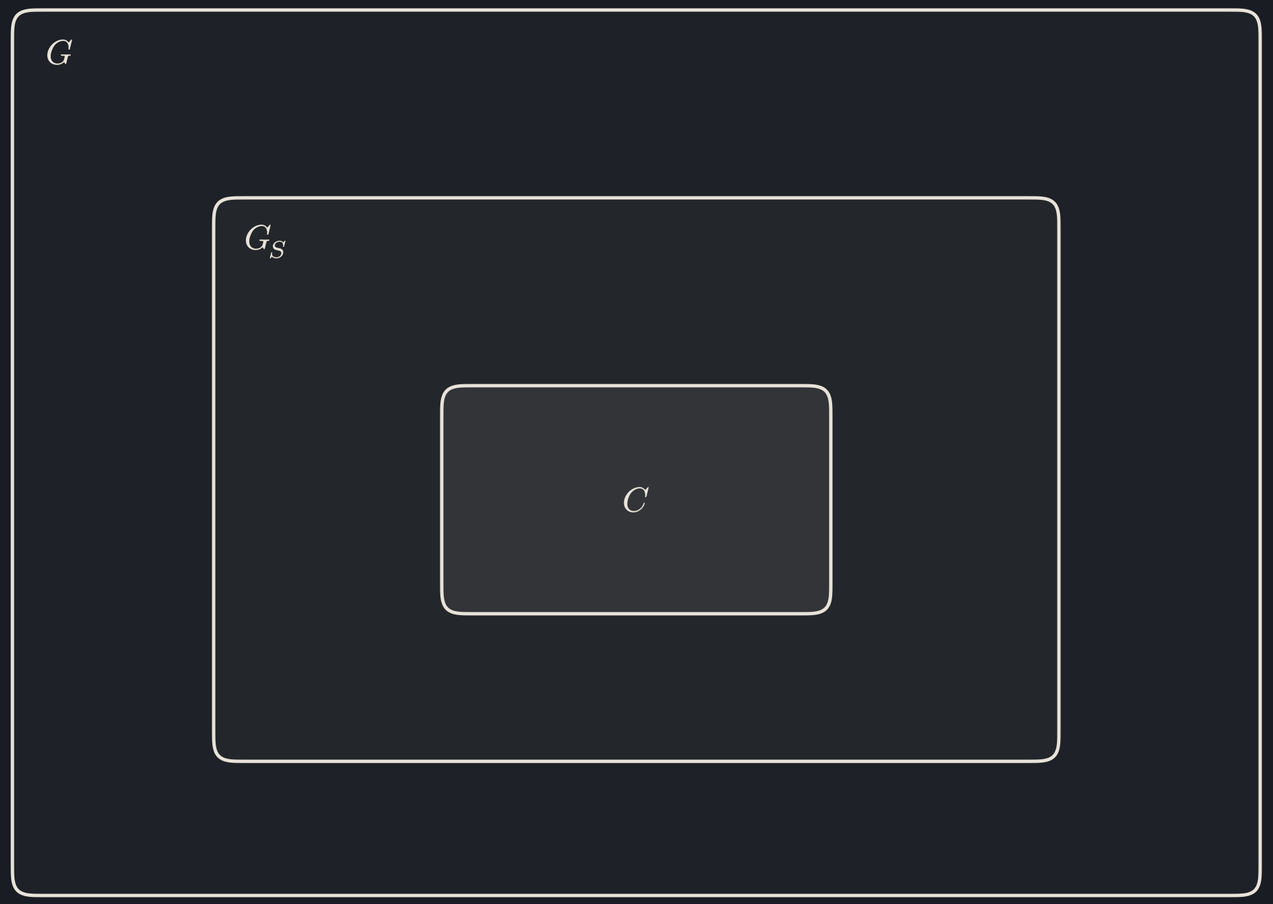
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This contradiction proves $G \notin$ SJ. □


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$G \notin$ SJ (Proposition 5.1)
Uncountably many indices: a pigeonhole on the defects
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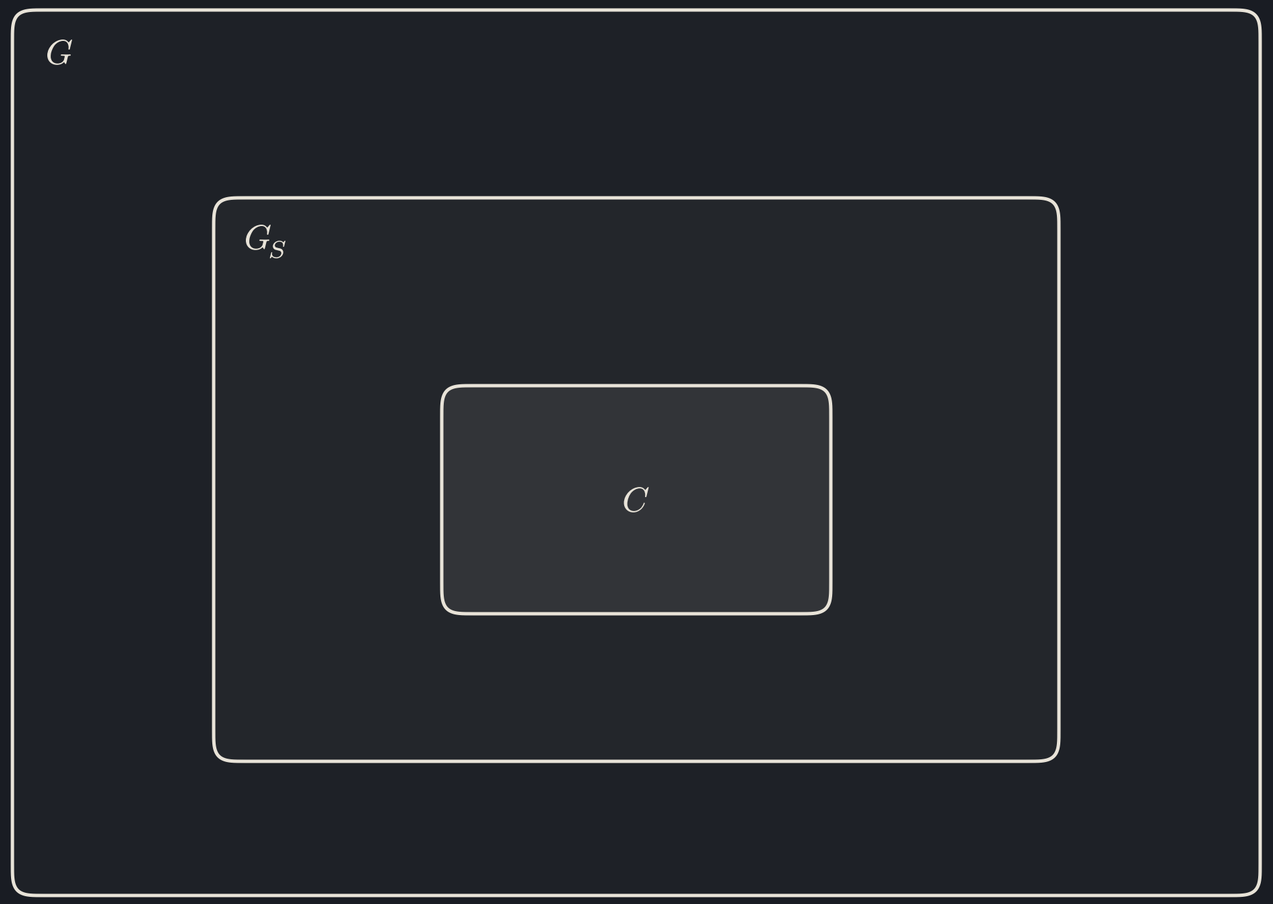


For a countable set $S \subseteq \omega_1$, put $E_S = \bigoplus_{\alpha \in S} \langle g_\alpha \rangle$ and let $G_S = \langle E_S,\ A_{\alpha\beta} : \alpha, \beta \in S,\ \alpha < \beta \rangle$.



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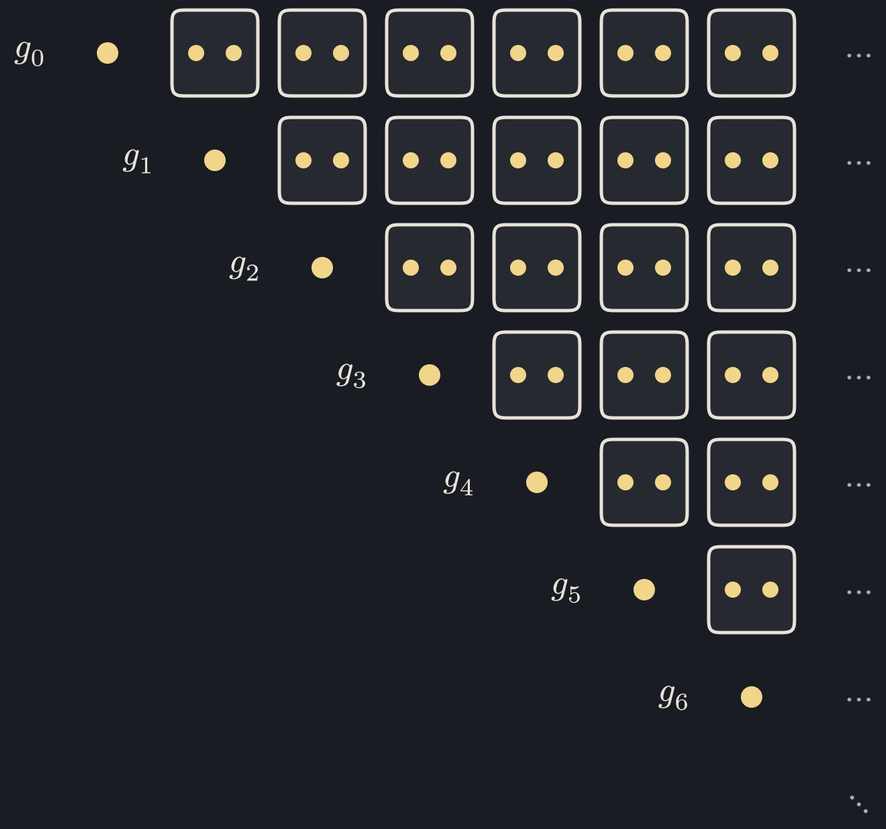
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Lemma 5.2. Every countable subgroup of G is contained in G_S for some countable set $S \subseteq \omega_1$.

We shall prove that $G_S \in \mathrm{SJ}$ for every countable S.



Goal: $G_S \in \mathrm{SJ}$

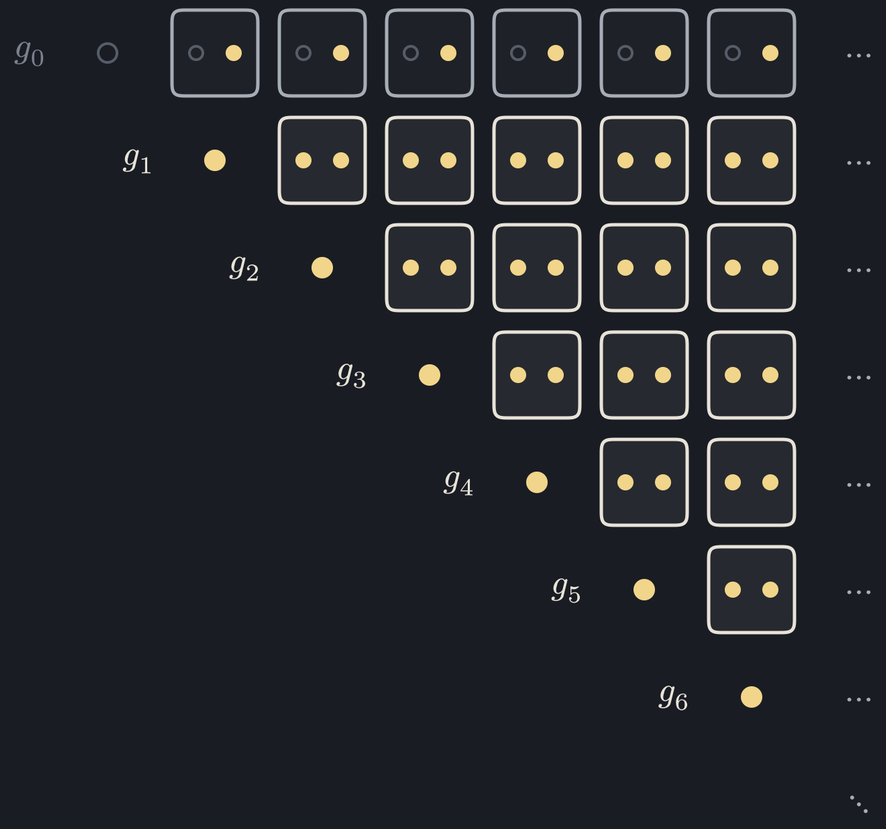
$m = 0$

Proof. Enumerate $S = \{s_0, s_1, s_2, ...\}$ and write $g_i = g_{s_i}$, $A_{ij} = A_{s_i s_j}$.



Row i, column j: the factor A_{ij}. Its dots are g_i (left) and g_j (right): golden while alive, empty once killed.

The chain of K_m



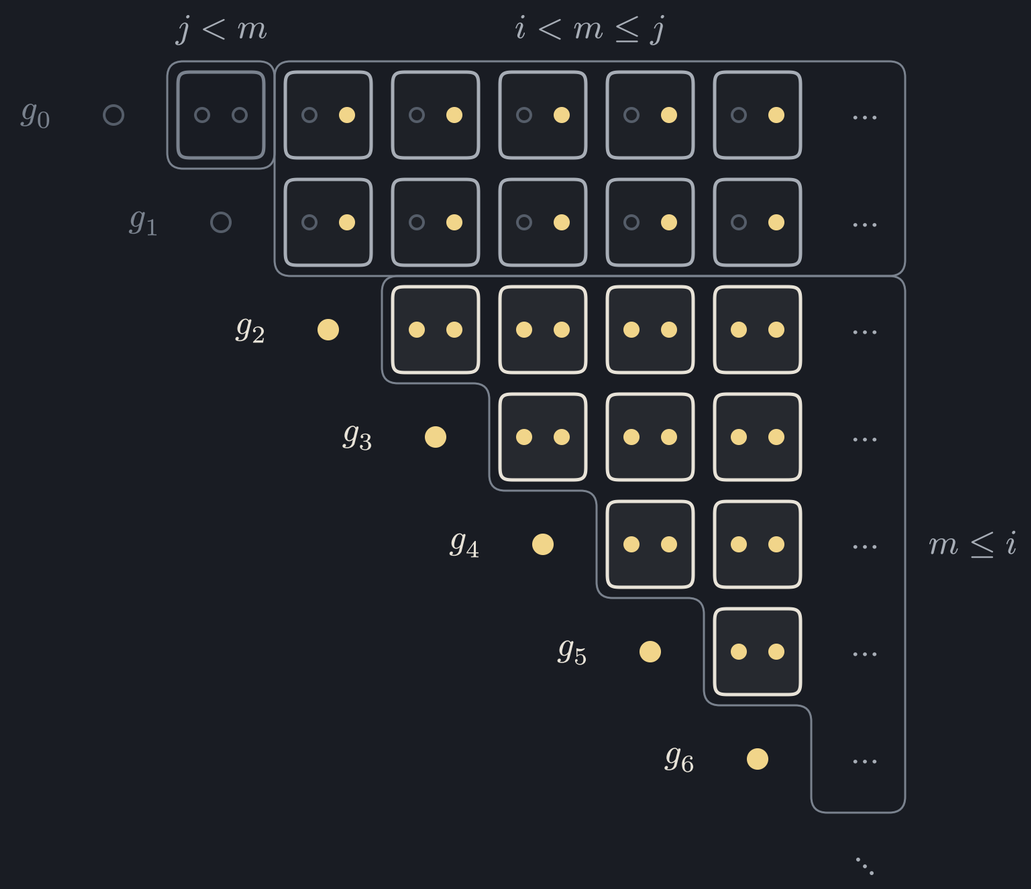
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$m = 1$

Kill the generators one at a time:

$$R_m = \langle g_i : i \in I,\ i < m \rangle^{G_S}, \qquad G_m = G_S / R_m$$

The chain of K_m



Goal: $G_S \in \mathrm{SJ}$

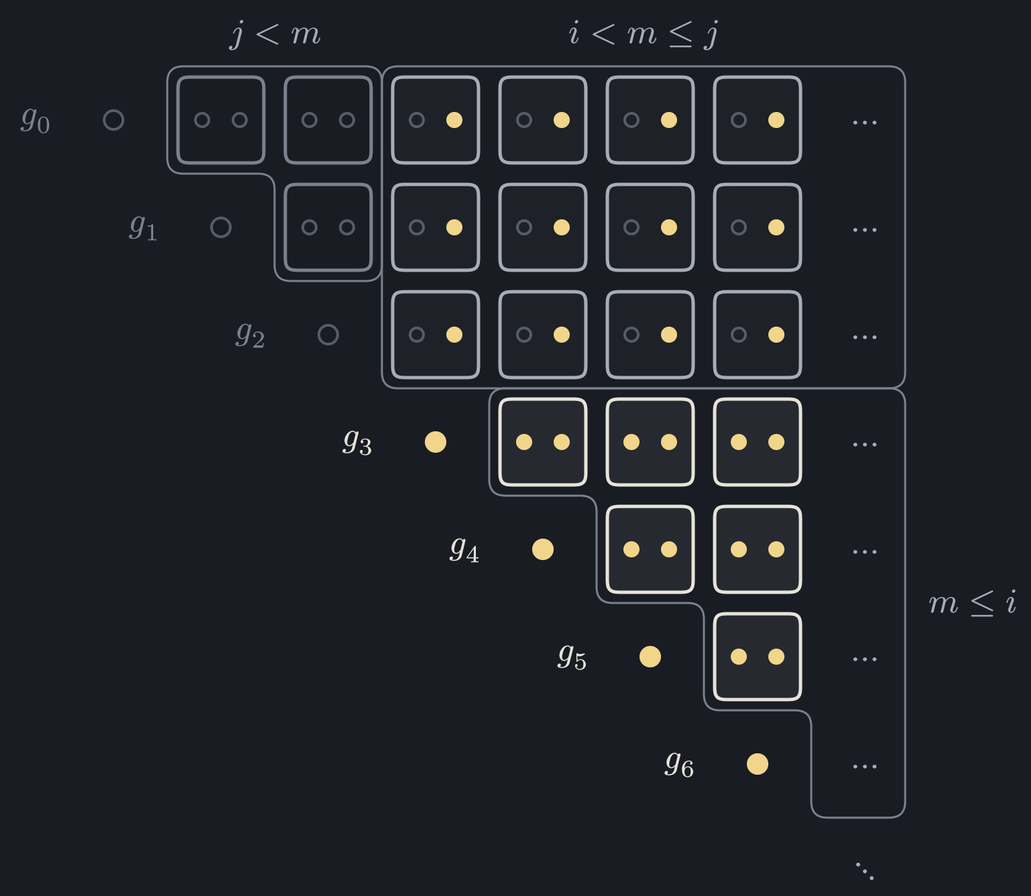
$m = 2$

● ●	$m \leq i$	A_{ij}
○ ●	$i < m \leq j$	A_{ij}/X_i^{ij}
○ ○	$j < m$	A_{ij}/N_{ij}

Identifications killed; the factor survives

Killing g_i kills $X_i^{ij} = \langle g_i \rangle^{A_{ij}}$ in each factor A_{ij}; killing g_i and g_j kills $N_{ij} = \langle g_i, g_j \rangle^{A_{ij}}$.

The chain of K_m



Goal: $G_S \in \mathrm{SJ}$

$m = 3$

● ●	$m \leq i$	A_{ij}
○ ●	$i < m \leq j$	A_{ij}/X_i^{ij}
○ ○	$j < m$	A_{ij}/N_{ij}

Identifications killed; the factor survives

In the middle case g_j survives: X_i^{ij} is not perfect while N_{ij} is, so $g_j \notin X_i^{ij}$ (Lemma 5.3).

So G_m is again an amalgam, of the surviving generators and the factors of the legend: the grid is a picture of G_m (Lemma 5.4).

Countable subgroups: factoring out the generators one at a time



Goal: $G_S \in \mathrm{SJ}$

$m = 3$

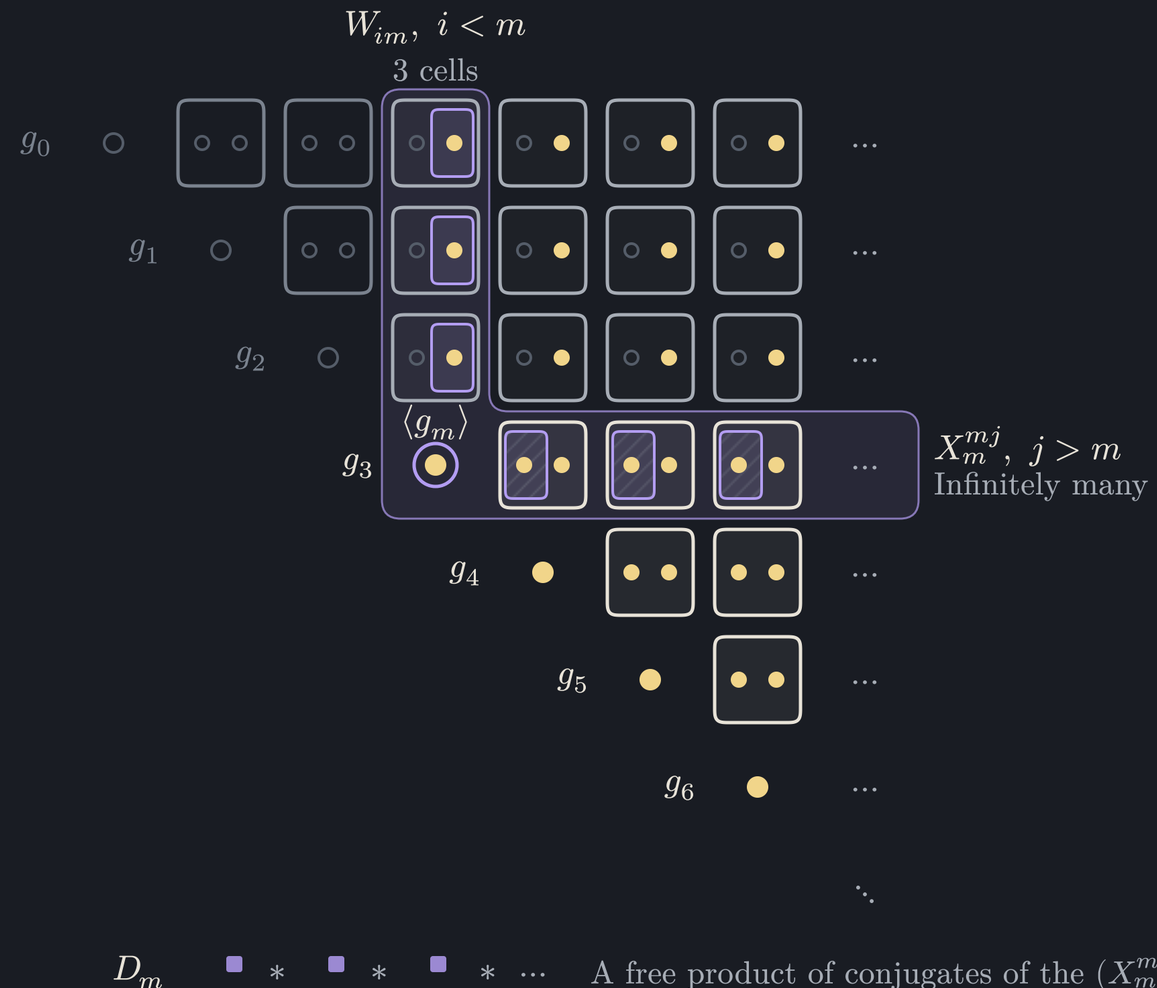
● ●	$m \le i$	A_{ij}
○ ●	$i < m \le j$	A_{ij}/X_i^{ij}
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Identifications killed; the factor survives

Now kill g_m. The kernel $K_m = R_{m+1}/R_m$ meets only the factors in the row and in the column of g_m: $\langle g_m \rangle$ itself, the X_m^{mj} in the row, and the $W_{im} = N_{im}/X_i^{im}$ in the column (Lemma 5.4).

The chain of K_m

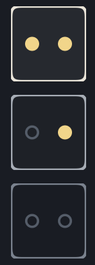
Countable subgroups: factoring out the generators one at a time



Goal: $G_S \in \mathrm{SJ}$

Lemma 5.5. $K_m \in \mathrm{SJ}$.

$m = 3$



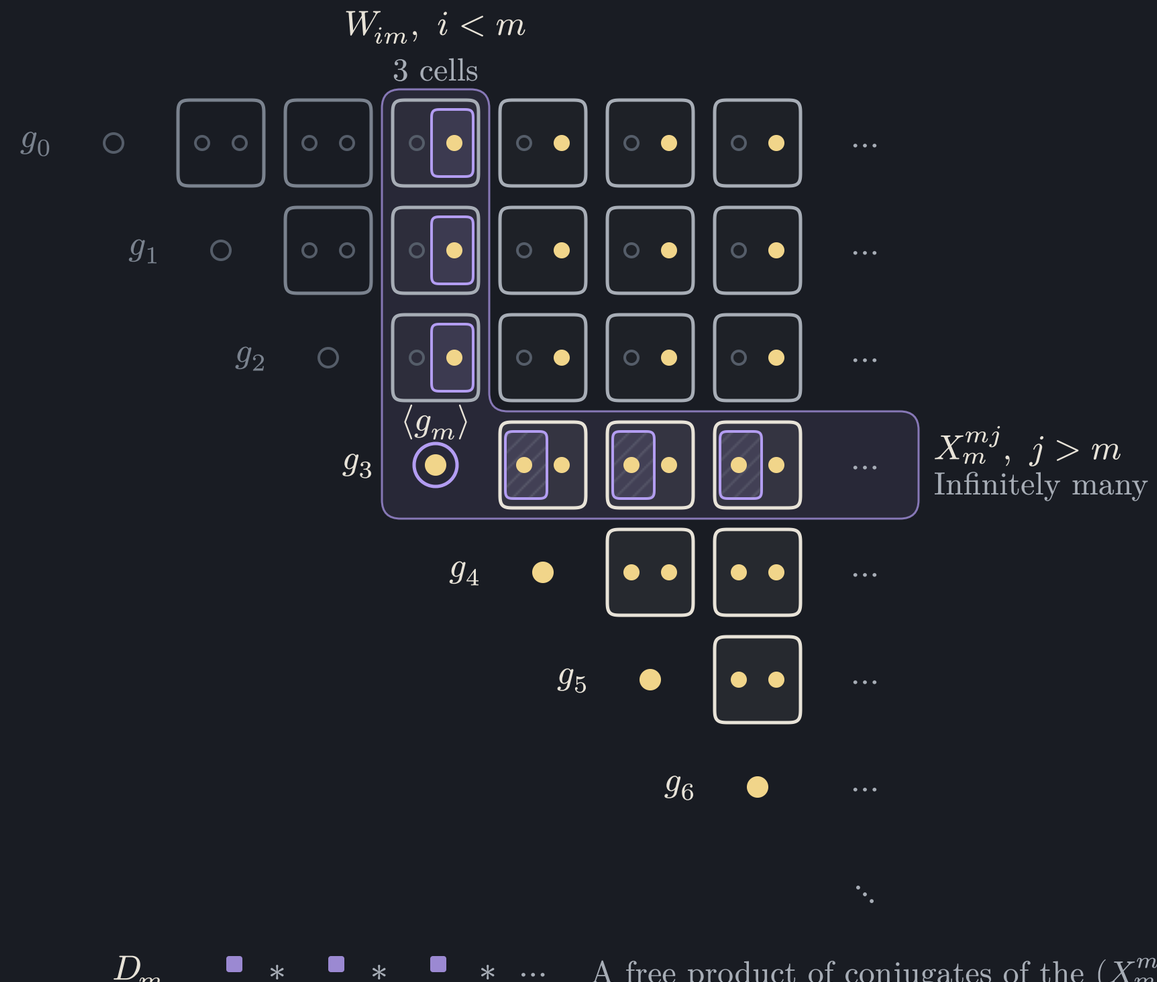
$m \le i$ A_{ij}

$i < m \le j$ A_{ij}/X_i^{ij}

$j < m$ A_{ij}/N_{ij}

Identifications killed; the factor survives

Proof. First factor out the derived subgroups $(X_m^{mj})'$: they have finite normal height, and the normal subgroup D_m they generate is a free product of their conjugates. So $D_m \in \mathrm{SJ}$.



Goal: $G_S \in \mathrm{SJ}$

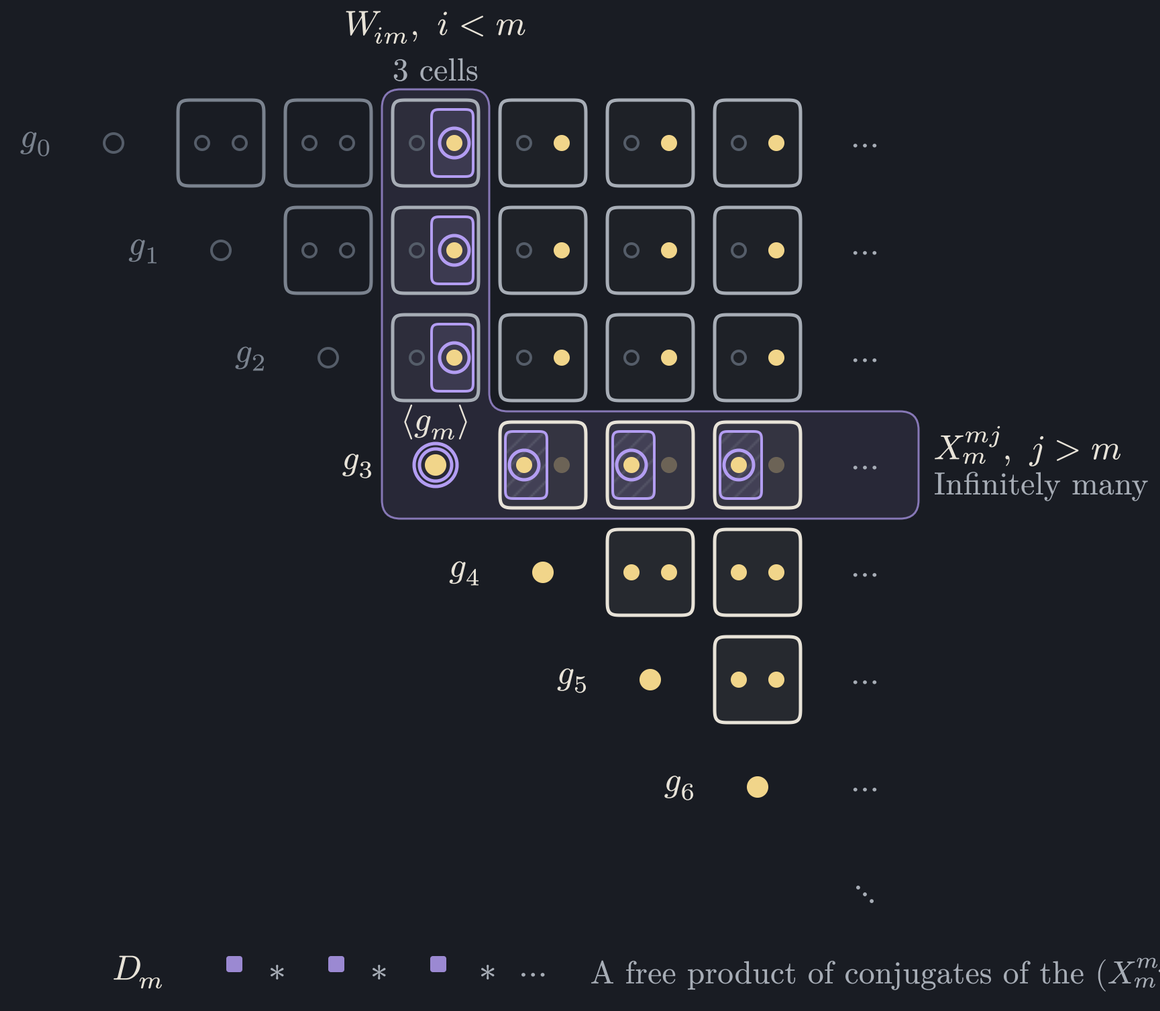
Lemma 5.5. $K_m \in \mathrm{SJ}$.

$m = 3$

- $\langle g_m \rangle \simeq C_2$
- $X_m^{mj}/(X_m^{mj})'$
  elementary abelian 2-groups
- W_{im}
  finite normal height
  finitely many

What is left of K_m is made of pieces of bounded height: $\langle g_m \rangle$, elementary abelian pieces on the row, and finitely many W_{im} on the column.

Countable subgroups: factoring out the generators one at a time

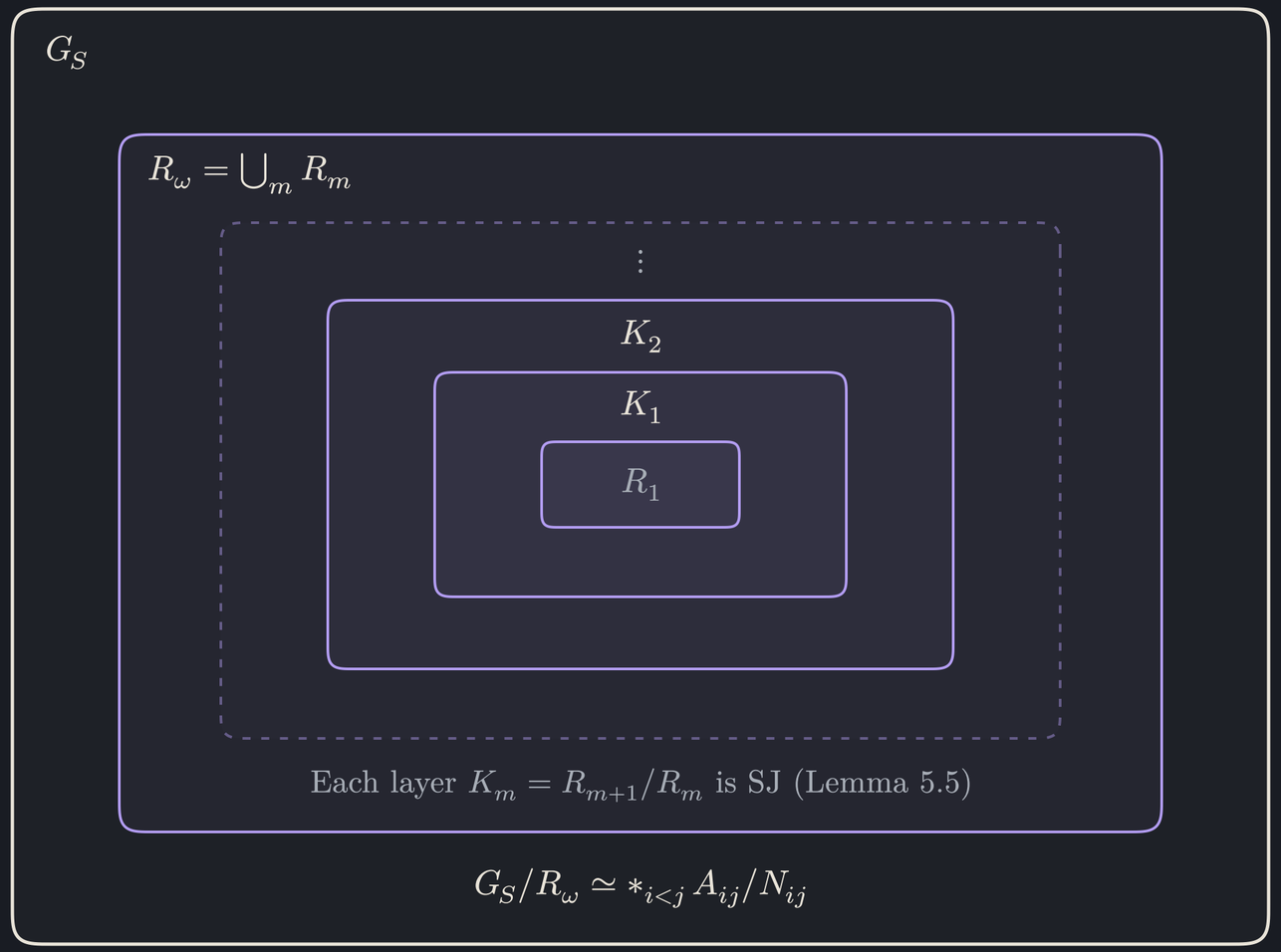


Goal: $G_S \in \mathrm{SJ}$

Lemma 5.5. $K_m \in \mathrm{SJ}$.

$m = 3$

They are glued only along $\langle g_m \rangle$, of order 2: by Theorem 4.4 they generate an SJ-group, with a free quotient on top. With $D_m \in \mathrm{SJ}$, this gives $K_m \in \mathrm{SJ}$. $\square$



Goal: $G_S \in \mathrm{SJ}$

$R_\omega = \bigcup_m R_m$ is built in layers $K_m = R_{m+1}/R_m \in \mathrm{SJ}$: stacking their SJ-series gives one for R_ω. So $R_\omega \in \mathrm{SJ}$.

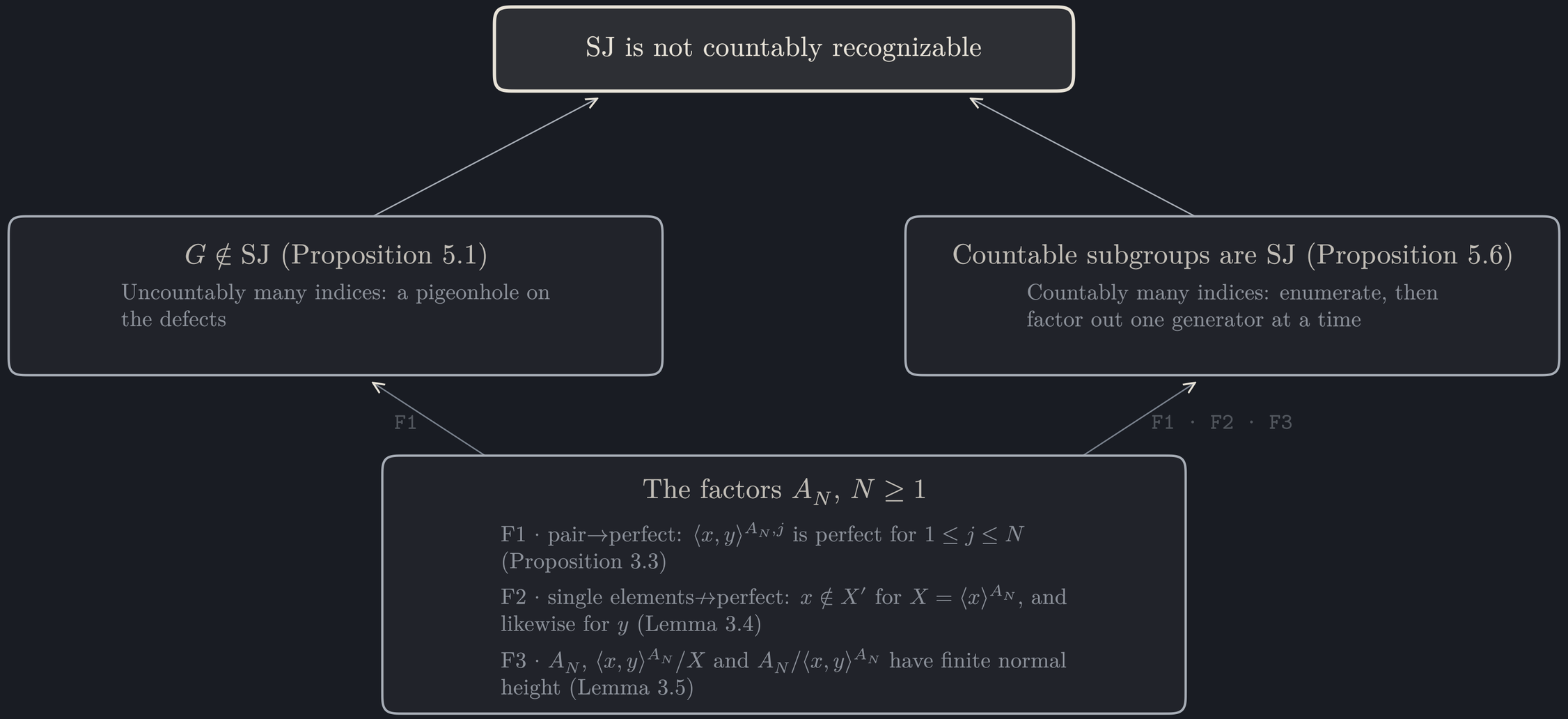
In G_S/R_ω every g_i is killed: it is the free product of the A_{ij}/N_{ij}, which have finite normal height. So $G_S/R_\omega \in \mathrm{SJ}$.

SJ is closed under extensions: $G_S \in \mathrm{SJ}$. $\square$

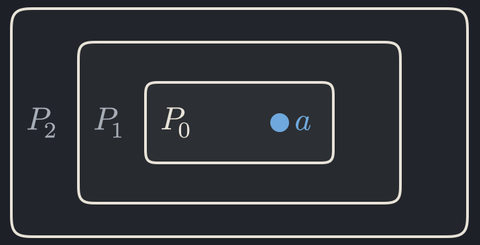
Proposition 5.6. Every countable subgroup of G belongs to SJ.

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Proof. Let $C \leq G$ be countable. By Lemma 5.2, there is a countable set $S \subseteq \omega_1$ such that $C \leq G_S$. Since $G_S \in \mathrm{SJ}$ and the class SJ is closed under taking subgroups, we have $C \in \mathrm{SJ}$. $\square$


SJ is not countably recognizable
$G \notin$ SJ (Proposition 5.1)
Uncountably many indices: a pigeonhole on the defects
Countable subgroups are SJ (Proposition 5.6)
Countably many indices: enumerate, then factor out one generator at a time
F1
F1 · F2 · F3
The factors A_N, $N \geq 1$
F1 · pair→perfect: $\langle x, y\rangle^{A_N, j}$ is perfect for $1 \leq j \leq N$ (Proposition 3.3)
F2 · single elements↛perfect: $x \notin X'$ for $X = \langle x\rangle^{A_N}$, and likewise for y (Lemma 3.4)
F3 · A_N, $\langle x, y\rangle^{A_N}/X$ and $A_N/\langle x, y\rangle^{A_N}$ have finite normal height (Lemma 3.5)

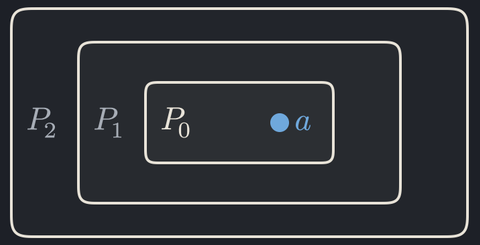
1 · One deep involution



$$P_0 = I(H) \rtimes H, \qquad P_{N+1} = T(P_N)$$

An involution a such that $\langle a \rangle^{P_N, j}$ is perfect for every $j \leq N$

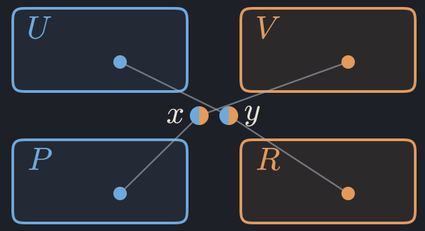
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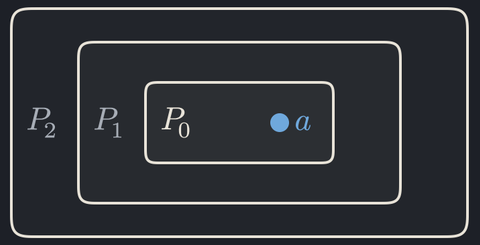
2 · The pair gives perfect, the singles do not



$$A_N = (U \oplus V) \rtimes (P \times R)$$

$\langle x, y \rangle^{A_N, j}$ is perfect for $j \leq N$, while $\langle x \rangle^{A_N}$ is not

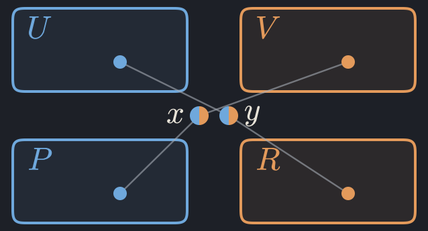
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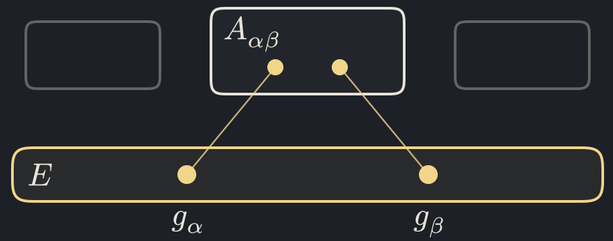
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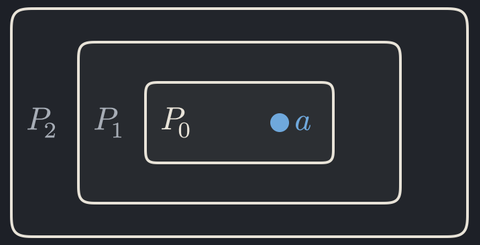
3 · Gluing along Klein fours



$$G = E * \mathop{*}_{\alpha<\beta<\omega_1} A_{\alpha\beta}$$

$A_{\alpha\beta} \cong A_{n(\alpha,\beta)}$, its pair x, y identified with $g_\alpha, g_\beta \in E$

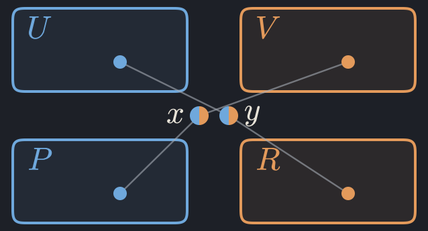
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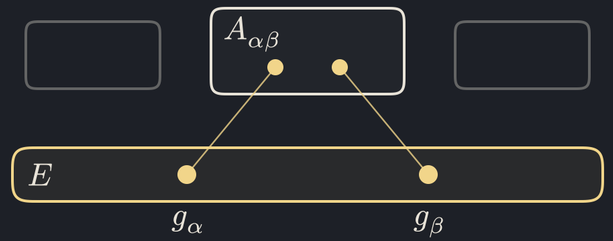
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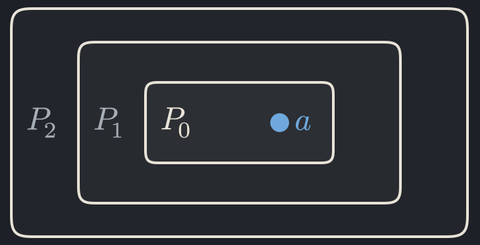
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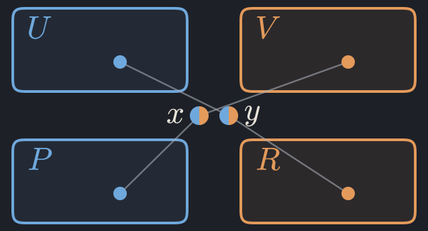
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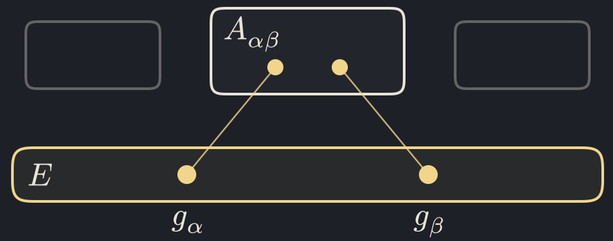
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Main Theorem. There exists a group G such that every countable subgroup of G belongs to SJ, whereas $G \notin \mathrm{SJ}$. Consequently the class SJ is not countably recognizable.

$[x_{rs}(u), x_{st}(v)] = x_{rt}(uv)$

$h_\lambda \longmapsto h_{\lambda m}$





$\langle x, y \rangle^{A_N, j}$

Thank you

$P_{N+1} = T(P_N)$

The group class SJ is not countably recognizable

$T(H) = W \rtimes M$

M. Brescia — B.G. Di Siena — E. Ingrosso — M. Trombetti



ω_1





$K^{G,n}$

$R_\omega = \bigcup_{m<\omega} R_m$

